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Trilocal Structures. IV. Momentum-Dependent Tree Functions

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Enlarging the set of tree functions to include those which depend on the momentum vector has the effect of introducing new families and subfamilies of functions. Four auxiliary conditions are used in the generation of these functions. These auxiliary conditions introduce, as eigenvalues, four parameters in terms of which the coefficients of the momentum-dependent functions can then be expressed as linear combinations of the 16 leading coefficients. These 16 are all rest-system coefficients. Thus the momentum-dependent part of the expansion is expressible in terms of the rest-system portion, using only these four parameters.

1. INTRODUCTION

The earlier articles in this set (Clapp et al., 1980, 1979, and 1981), which will be referred to here as I, II, and III, introduced notation, equations, and expansion functions for the trilocal system. The functions defined in II and III, however, were specialized to the rest system and did not involve the momentum vector **k**. The full Hamiltonian includes terms depending on **k**, included for example in the grouping H_k in (III.2.11b),[1] and this "momentum Hamiltonian" will introduce momentum-dependent expansion functions when it acts upon any of the momentum-independent functions in II or in III.

In particular, operation by H_k upon the tree functions in III will generate momentum-dependent tree functions, each with a factor $j_{m,n}$ containing the "radial" dependence. These functions group into families, as will be evident from the following sections.

[1]Equations from the earlier articles are cited with the paper number preceding the equation number.

Three new auxiliary operators will be introduced, each involving the momentum **k** and one or both of the relative gradient vectors, ∇_r and ∇_ρ. Requiring that these new operators be conserved will permit the coefficients of the momentum-dependent functions to be expressed as linear combinations of the first 16 C_j, through formulas similar to those given in Appendix E of III.

In the articles to follow this one, the interconnecting relationships will be used to reduce an infinite system of coupled linear equations to a finite system in which the matrix elements are functions of the eigenvalues of these conserved operators. The finite system has an equally finite set of discrete solutions, whose masses are to be compared with the observed lepton masses.

But that is for the articles to follow. The present article has the sufficiently large chore of sorting out the momentum-dependent tree functions.

2. SPIN FUNCTIONS

For the rest system, there are 16 σ-spin functions that enter into the function expansions. These 16 were listed in (II.4.1) and are given again here:

$$\begin{gathered} {}^{2b}(1) \quad {}^{2c}(1) \quad {}^{2b}(\mathbf{r}) \quad {}^{2c}(\mathbf{r}) \quad {}^{2b}(\boldsymbol{\rho}) \quad {}^{2c}(\boldsymbol{\rho}) \quad {}^{4}(\mathbf{r}) \quad {}^{4}(\boldsymbol{\rho}) \\ {}^{2b}(i\mathbf{r}\times\boldsymbol{\rho}) \quad {}^{2c}(i\mathbf{r}\times\boldsymbol{\rho}) \quad {}^{4}(i\mathbf{r}\times\boldsymbol{\rho}) \quad {}^{4}(\mathbf{rr}) \quad {}^{4}(\boldsymbol{\rho\rho}) \\ {}^{4}(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r}) \quad {}^{4}(i\mathbf{rr}\times\boldsymbol{\rho}) \quad {}^{4}(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho}) \end{gathered} \tag{1}$$

These are mnemonics for functions which are given explicitly in Clapp (1961). As can be seen from the notation above, they include two ${}^2S$ functions, four ${}^2P$ functions of odd parity, two ${}^4P$ functions of odd parity, two ${}^2P$ functions of even parity, one ${}^4P$ function of even parity, three ${}^4D$ functions of even parity, and two ${}^4D$ functions of odd parity. There are altogether eight σ-spin functions of even parity in this list, and eight σ-spin functions of odd parity.

When the operator H_k, defined by

$$H_k = (1/9)(\boldsymbol{\sigma}^s\cdot\mathbf{k} - 3P^\tau\boldsymbol{\sigma}^b\cdot\mathbf{k} + \boldsymbol{\sigma}^c\cdot\mathbf{k}) \tag{2}$$

acts upon expansion functions containing these σ-spin functions, and acts again upon the resulting functions, 18 more σ-spin functions are generated. Each of these involves the vector **k**, either linearly or quadratically. The mnemonics chosen for these eighteen are

$$
\begin{gathered}
{}^{2b}(\mathbf{k}) \quad {}^{2c}(\mathbf{k}) \quad {}^{4}(\mathbf{k}) \quad {}^{2b}(i\mathbf{k}\times\mathbf{r}) \quad {}^{2c}(i\mathbf{k}\times\mathbf{r}) \quad {}^{2b}(i\mathbf{k}\times\boldsymbol{\rho}) \\
{}^{2c}(i\mathbf{k}\times\boldsymbol{\rho}) \quad {}^{4}(i\mathbf{k}\times\mathbf{r}) \quad {}^{4}(i\mathbf{k}\times\boldsymbol{\rho}) \quad {}^{4}(\mathbf{kk}) \quad {}^{4}(\mathbf{kr}+\mathbf{rk}) \quad {}^{4}(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k}) \\
{}^{4}(i\mathbf{kr}\times\boldsymbol{\rho}) \quad {}^{4}(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r}) \quad {}^{4}(i\mathbf{rk}\times\mathbf{r}) \quad {}^{4}(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho}) \\
{}^{4}(i\mathbf{kk}\times\mathbf{r}) \quad {}^{4}(i\mathbf{kk}\times\boldsymbol{\rho})
\end{gathered}
\tag{3}
$$

The explicit functions, for most of these 18, can be obtained from functions in (1) through simple substitution of one vector for another. Two only, of those in (3), need special attention here. One of these, ${}^{4}(i\mathbf{kr}\times\boldsymbol{\rho})$, has the explicit components

$$
{}^{4}(i\mathbf{kr}\times\boldsymbol{\rho})^{+1/2}_{1/2} = \begin{array}{|c|}
\hline
-(k_z z-\mathbf{k}\cdot\mathbf{r}/2)(\rho_x-i\rho_y)+(k_z\rho_z-\mathbf{k}\cdot\boldsymbol{\rho}/2)(x-iy) \\
\hline
i[k_z(x\rho_y-y\rho_x)-\mathbf{k}\cdot\mathbf{r}\times\boldsymbol{\rho}/3] \\
i[k_z(x\rho_y-y\rho_x)-\mathbf{k}\cdot\mathbf{r}\times\boldsymbol{\rho}/3] \\
i[k_z(x\rho_y-y\rho_x)-\mathbf{k}\cdot\mathbf{r}\times\boldsymbol{\rho}/3] \\
\hline
-(k_z z-\mathbf{k}\cdot\mathbf{r}/2)(\rho_x+i\rho_y)+(k_z\rho_z-\mathbf{k}\cdot\boldsymbol{\rho}/2)(x+iy) \\
-(k_z z-\mathbf{k}\cdot\mathbf{r}/2)(\rho_x+i\rho_y)+(k_z\rho_z-\mathbf{k}\cdot\boldsymbol{\rho}/2)(x+iy) \\
-(k_z z-\mathbf{k}\cdot\mathbf{r}/2)(\rho_x+i\rho_y)+(k_z\rho_z-\mathbf{k}\cdot\boldsymbol{\rho}/2)(x+iy) \\
\hline
(k_x+ik_y)(x+iy)\rho_z-(k_x+ik_y)z(\rho_x+i\rho_y) \\
\hline
\end{array}^{\,\sigma}
\tag{4a}
$$

$$
{}^{4}(i\mathbf{kr}\times\boldsymbol{\rho})^{-1/2}_{1/2} = \begin{array}{|c|}
\hline
(k_x-ik_y)(x-iy)\rho_z-(k_x-ik_y)z(\rho_x-i\rho_y) \\
\hline
(k_z z-\mathbf{k}\cdot\mathbf{r}/2)(\rho_x-i\rho_y)-(k_z\rho_z-\mathbf{k}\cdot\boldsymbol{\rho}/2)(x-iy) \\
(k_z z-\mathbf{k}\cdot\mathbf{r}/2)(\rho_x-i\rho_y)-(k_z\rho_z-\mathbf{k}\cdot\boldsymbol{\rho}/2)(x-iy) \\
(k_z z-\mathbf{k}\cdot\mathbf{r}/2)(\rho_x-i\rho_y)-(k_z\rho_z-\mathbf{k}\cdot\boldsymbol{\rho}/2)(x-iy) \\
\hline
-i[k_z(x\rho_y-y\rho_x)-\mathbf{k}\cdot\mathbf{r}\times\boldsymbol{\rho}/3] \\
-i[k_z(x\rho_y-y\rho_x)-\mathbf{k}\cdot\mathbf{r}\times\boldsymbol{\rho}/3] \\
-i[k_z(x\rho_y-y\rho_x)-\mathbf{k}\cdot\mathbf{r}\times\boldsymbol{\rho}/3] \\
\hline
(k_z z-\mathbf{k}\cdot\mathbf{r}/2)(\rho_x+i\rho_y)-(k_z\rho_z-\mathbf{k}\cdot\boldsymbol{\rho}/2)(x+iy) \\
\hline
\end{array}^{\,\sigma}
\tag{4b}
$$

The explicit components for the other, ${}^{4}(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})$, can be obtained directly from permuted versions of (4). There is no third such function, as a result of the easily verified identity

$$
{}^{4}(i\mathbf{kr}\times\boldsymbol{\rho})+{}^{4}(i\mathbf{r}\boldsymbol{\rho}\times\mathbf{k})+{}^{4}(i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})=0 \tag{5}
$$

The choice of two combinations of these three, as the independent functions to be used, is of course arbitrary. Considerations of symmetry were paramount, in the actual choices that have been made here. One of the chosen functions changes sign when $\mathbf{r}$ and $\boldsymbol{\rho}$ are interchanged, while the other remains unchanged.

Of the 18 σ-spin functions listed in (3), only 16 are truly independent. This follows from two further identities, less obvious than (5) but nevertheless readily verified. These identities are

$$
\begin{aligned}
&{}^4(i\mathbf{r}\mathbf{r}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho})-{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})\ -{}^4(i\mathbf{r}\mathbf{k}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})\\
&\qquad\qquad\qquad +{}^4(i\mathbf{r}\mathbf{k}\times\mathbf{r})\rho^2+{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})r^2=0 \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
&{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2]-{}^4(\mathbf{r}\mathbf{r})[(\mathbf{k}\cdot\boldsymbol{\rho})^2-k^2\rho^2]\\
&\quad -{}^4(\boldsymbol{\rho}\boldsymbol{\rho})[(\mathbf{k}\cdot\mathbf{r})^2-k^2r^2]+{}^4(\mathbf{k}\mathbf{r}+\mathbf{r}\mathbf{k})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})\rho^2]\\
&\quad +{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})r^2]-{}^4(\mathbf{k}\mathbf{k})[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2]=0 \qquad (7)
\end{aligned}
$$

These identities reduce the functions in (3) to eight independent functions of even parity and eight independent functions of odd parity, the same count as in the rest-system set in (1).

It is not very convenient to attempt to remove two of the functions in (3), since (6) and (7) are not easily solved for an individual function in terms of other functions. Accordingly, all 18 of the σ-spin functions in (3) will be used in the construction of expansion functions, but when the combinations (6) and (7) arise, it will be remembered that they are identically zero.

With this cautionary note, we can proceed to construct momentum-dependent tree functions. The σ-spin functions that are contained will come from the lists in (1) and (3). There will be τ-spin functions in each expansion function, but no new τ-spin functions are introduced by the operator H_k in (2), so that the τ-spin functions that will appear are just $(+)^\tau$ and $(-)^\tau$, as defined in (III.2.2).

3. AUXILIARY OPERATORS

The auxiliary operator $P^\tau(\nabla_r\cdot\nabla_\rho)$ was introduced earlier through the operator equation (III.3.1), given again here as

$$P^\tau(\nabla_r\cdot\nabla_\rho)\Phi=-\kappa_r\kappa_\rho\nu\Phi \qquad (8)$$

This auxiliary condition also introduces the eigenvalue ν, which was earlier of use in providing expressions for the coefficients of the rest-system expansion functions, as shown in Appendices E and F of III.

The generalization to momentum-dependent functions opens the way to the introduction of two similar operator equations:

$$(i\mathbf{k}\cdot\nabla_r)\Phi = k\kappa_r\nu'\Phi \tag{9}$$

$$P^\tau(i\mathbf{k}\cdot\nabla_\rho)\Phi = k\kappa_\rho\nu''\Phi \tag{10}$$

These are auxiliary conditions that introduce the eigenvalues ν' and ν'', analogous to ν to some degree.

It is found that the operators in (9) and (10) have matrix representations which are symmetrical, when the function system is developed in such a way that the momentum Hamiltonian H_k in (2) has a symmetrical matrix representation. The operator H_k couples across families, while the operators (8)–(10) are more restrictive, coupling functions which are within limited families. The latter operators are thus particularly useful in developing the sets of functions which are denoted as families, and in providing reduction formulas permitting their coefficients to be expressed simply in terms of a limited, finite number of leading coefficients, belonging to a similarly limited number of leading functions in the separate families.

There is also another auxiliary operator of particular value, which satisfies the operator equation

$$P^\tau(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\Phi = -k\kappa_r\kappa_\rho\gamma\Phi \tag{11}$$

The matrix representation of this operator is not symmetric, but is skew symmetric. This is important in practical utilization of (11), but does not diminish the utility of the auxiliary condition (11) in any way. Its value comes from the coupling which it provides between families which are not coupled by the simpler operators in (8)–(10).

The square of the operator in (11) can be expressed as a function of the operators in (8)–(10). For this we can use the vector identity

$$\begin{aligned}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})^2 = {} & 2(\mathbf{A}\cdot\mathbf{B})(\mathbf{B}\cdot\mathbf{C})(\mathbf{C}\cdot\mathbf{A}) - A^2(\mathbf{B}\cdot\mathbf{C})^2 - B^2(\mathbf{C}\cdot\mathbf{A})^2 \\ & - C^2(\mathbf{A}\cdot\mathbf{B})^2 + A^2B^2C^2\end{aligned} \tag{12}$$

which is valid for any three vectors **A**, **B**, **C**. From this we can establish that

$$\gamma^2 = -2\nu\nu'\nu'' + \nu^2 + \nu'^2 + \nu''^2 - 1 \tag{13}$$

This fixes the magnitude of γ, once we know ν, ν', and ν'', but we are still free to choose the sign of γ, and this will prove to be important.

4. ^2S-STATE FAMILIES

When the operator in (9) is allowed to act on the initial rest-system expansion function

$$\varphi_1 = N_0 j_{0,0}\left[(+)^{\tau 2b}(1)+(-)^{\tau 2c}(1)\right] \tag{14}$$

it generates a new function which has the form

$$\varphi_1^{1,0,k} = (iN_0/k)(3)^{1/2}\kappa_r j_{1,0}(\mathbf{k}\cdot\mathbf{r})\left[(+)^{\tau 2b}(1)+(-)^{\tau 2c}(1)\right] \tag{15}$$

As in previous work, the normalization constant in (15) is not known until the operator in (9) is allowed to act again, regenerating the starting function together with further functions. Requiring that the matrix representation of the operator be symmetrical then fixes the normalization constant.

In a similar way, the use of the operator in (10) generates the function

$$\varphi_1^{0,1,k} = (iN_0/k)(3)^{1/2}\kappa_\rho j_{0,1}(\mathbf{k}\cdot\boldsymbol{\rho})\left[(-)^{\tau 2b}(1)+(+)^{\tau 2c}(1)\right] \tag{16}$$

in which the τ-spin functions have been altered by the operator P^τ in (10). Further action by these operators generates the functions

$$\begin{aligned}\varphi_1^{2,0,2k} = {} & (N_0/k^2)(45/4)^{1/2}\kappa_r^2 j_{2,0}\left[(\mathbf{k}\cdot\mathbf{r})^2-k^2r^2/3\right]\\ & \times\left[(+)^{\tau 2b}(1)+(-)^{\tau 2c}(1)\right]\end{aligned} \tag{17}$$

$$\begin{aligned}\varphi_1^{0,2,2k} = {} & (N_0/k^2)(45/4)^{1/2}\kappa_\rho^2 j_{0,2}\left[(\mathbf{k}\cdot\boldsymbol{\rho})^2-k^2\rho^2/3\right]\\ & \times\left[(+)^{\tau 2b}(1)+(-)^{\tau 2c}(1)\right]\end{aligned} \tag{18}$$

$$\begin{aligned}\varphi_1^{1,1,2k} = {} & (N_0/k^2)(27/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-k^2(\mathbf{r}\cdot\boldsymbol{\rho})/3\right]\\ & \times\left[(-)^{\tau 2b}(1)+(+)^{\tau 2c}(1)\right]\end{aligned} \tag{19}$$

and many more.

The operator in (11), acting on the function in (14), generates the function

$$\begin{aligned}\varphi_1^{1,1,k} = {} & (N_0/k)(9/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}(i\mathbf{k}\cdot\mathbf{r}\times\boldsymbol{\rho})\\ & \times\left[(-)^{\tau 2b}(1)+(+)^{\tau 2c}(1)\right]\end{aligned} \tag{20}$$

and action by (9) and (10) upon (20) then generates functions resembling (15)–(19) but all containing the factor $(i\mathbf{k}\cdot\mathbf{r}\times\boldsymbol{\rho})$.

Similar sets of functions can be generated as a result of operations upon other starting functions, such as

$$\varphi_{17}=N_0(3)^{1/2}\kappa_r\kappa_\rho j_{1,1}(\mathbf{r}\cdot\boldsymbol{\rho})\big[(-)^{\tau 2b}(1)+(+)^{\tau 2c}(1)\big] \tag{21}$$

These include

$$\varphi_1^{2,1,k}=(iN_0/k)(27/2)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}\big[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})r^2/3\big]$$
$$\times\big[(-)^{\tau 2b}(1)+(+)^{\tau 2c}(1)\big] \tag{22}$$

$$\varphi_1^{2,2,2k}=\big(N_0/k^2\big)(2025/14)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}$$
$$\times\big[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})^2\rho^2/3-(\mathbf{k}\cdot\boldsymbol{\rho})^2 r^2/3$$
$$-(\mathbf{r}\cdot\boldsymbol{\rho})^2k^2/3+2k^2r^2\rho^2/9\big]\big[(+)^{\tau 2b}(1)+(-)^{\tau 2c}(1)\big] \tag{23}$$

For each function generated from φ_1 or φ_{17}, there will be a corresponding function generated from φ_2 or φ_{18}, defined in Appendix A of III:

$$\varphi_2=N_0(1/3)^{1/2}j_{0,0}\big[3(+)^{\tau 2b}(1)-(-)^{\tau 2c}(1)\big] \tag{24}$$

$$\varphi_{18}=N_0\kappa_r\kappa_\rho j_{1,1}(\mathbf{r}\cdot\boldsymbol{\rho})\big[3(-)^{\tau 2b}(1)-(+)^{\tau 2c}(1)\big] \tag{25}$$

In each case the change in the grouping of spin functions is accompanied by a change in the normalization constant, which is multiplied by $(1/3)^{1/2}$.

When the expanded wave function

$$\Phi=C_1\varphi_1+C_2\varphi_2+\cdots+C_1^{1,0,k}\varphi_1^{1,0,k}+\cdots \tag{26}$$

is substituted into the operator relationships (8)–(11), and terms involving each function are collected separately, the result is relationships among the coefficients. These relationships can be solved to give the higher coefficients in terms of lower ones. In particular, we find

$$C_{17}=-3^{1/2}\nu C_1 \tag{27}$$

$$C_1^{1,0,k}=-3^{1/2}\nu' C_1 \tag{28}$$

$$C_1^{0,1,k}=-3^{1/2}\nu'' C_1 \tag{29}$$

$$C_1^{1,1,k} = (9/2)^{1/2} \gamma C_1 \tag{30}$$

$$C_1^{2,0,2k} = -(45/4)^{1/2} [\nu'^2 - 1/3] C_1 \tag{31}$$

$$C_1^{0,2,2k} = -(45/4)^{1/2} [\nu''^2 - 1/3] C_1 \tag{32}$$

$$C_1^{1,1,2k} = -(27/2)^{1/2} [\nu'\nu'' - \nu/3] C_1 \tag{33}$$

$$C_1^{2,1,k} = (27/2)^{1/2} [\nu\nu' - \nu''/3] C_1 \tag{34}$$

$$C_1^{2,2,2k} = (2025/14)^{1/2} [\nu\nu'\nu'' - \nu^2/3 - \nu'^2/3 - \nu''^2/3 + 2/9] C_1 \tag{35}$$

with similar reduction equations for C_{18} and $C_2^{1,0,k}$, and so forth, expressed as multiples of C_2.

Many of the functions, such as (17) and (18), can be seen to have the form of Legendre polynomials insofar as their dependence upon a cosine is involved. This is reflected in the corresponding coefficients, as shown in (31) and (32), where the parameters ν' and ν'' play the role of cosines. Others of the set of functions, for example (22) and (23), and the corresponding coefficients, (34) and (35), contain what are evidently generalizations of Legendre polynomials. Three different cosines are involved, but they are not fully independent: when two cosines are close to unity, the third must also be fairly close to unity since the three vectors, **k**, **r**, and $\boldsymbol{\rho}$, are constrained in this case to be nearly parallel.

These functions of three cosines are hyperspherical harmonics in nine dimensions, but for a particular kind of expansion in which there are three "radial" scalars, the magnitudes of the three vectors. A different system of hyperspherical harmonics arises when the variables for the "bowl" expansion of II are selected.

5. ^4P-STATE FAMILIES

Within the first 16 rest-system tree functions, there are three quartet P-state functions, two having odd parity,

$$\varphi_3 = iN_0(2/3)^{1/2} \kappa_r j_{1,0}(-)^{\tau 4}(\mathbf{r}) \tag{36}$$

$$\varphi_4 = iN_0(2/3)^{1/2} \kappa_\rho j_{0,1}(+)^{\tau 4}(\boldsymbol{\rho}) \tag{37}$$

the third having even parity:

$$\varphi_9 = N_0 \kappa_r \kappa_\rho j_{1,1}(+)^{\tau 4}(i\mathbf{r} \times \boldsymbol{\rho}) \tag{38}$$

When the operators in (8)–(10) act on these three functions, the result is two separate families of momentum-dependent functions. However, the operator in (11), itself having odd parity, couples across between the odd-parity family and the even-parity family, joining them into a single system.

The structure of these families is sufficiently complicated to make notational difficulties, and the notational choices that have been made here should be considered as very tentative. One function that appears immediately, when the operators in (9) and (10) are applied to (36) and (37), is the following, which has been given a nonideal name:

$$\varphi_3^{0,0,k} = (N_0/k)(2/3)^{1/2} j_{0,0}(-)^{\tau\,4}(\mathbf{k}) \tag{39}$$

Other functions which appear at the same time are

$$\varphi_3^{2,0,k} = (N_0/k)(3)^{1/2} \kappa_r^2 j_{2,0}(-)^{\tau}\left[{}^4(\mathbf{r})(\mathbf{k}\cdot\mathbf{r}) - {}^4(\mathbf{k})r^2/3\right] \tag{40}$$

$$\varphi_3^{0,2,k} = (N_0/k)(3)^{1/2} \kappa_\rho^2 j_{0,2}(-)^{\tau}\left[{}^4(\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho}) - {}^4(\mathbf{k})\rho^2/3\right] \tag{41}$$

The above two are straightforward, but the following three were not immediately or easily sorted out:

$$\varphi_{3s}^{1,1,k} = (N_0/k)(2/5)^{1/2} \kappa_r \kappa_\rho j_{1,1}(+)^{\tau}$$

$$\times\left[{}^4(\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho}) + {}^4(\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r}) + {}^4(\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})\right] \tag{42}$$

$$\varphi_{3b}^{1,1,k} = (N_0/k)(3/2)^{1/2} \kappa_r \kappa_\rho j_{1,1}(+)^{\tau}$$

$$\times\left[{}^4(\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho}) - {}^4(\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})\right] \tag{43}$$

$$\varphi_{3c}^{1,1,k} = (N_0/k)(1/2)^{1/2} \kappa_r \kappa_\rho j_{1,1}(+)^{\tau}$$

$$\times\left[{}^4(\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho}) + {}^4(\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r}) - {}^4(\mathbf{k})2(\mathbf{r}\cdot\boldsymbol{\rho})\right] \tag{44}$$

Among the alternative choices, only these three linear combinations will give symmetry to the pertinent matrix elements in the matrix representations of the operators in (8)–(10), and the appropriate skew-symmetry to the matrix form of the operator in (11).

Further operations, particularly with the operator in (8), give the relatively straightforward functions

$$\varphi_3^{3,1,k} = (N_0/k)(15)^{1/2}\kappa_r^3\kappa_\rho j_{3,1}(+)^\tau$$

$$\times\{{}^4(\mathbf{r})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\boldsymbol{\rho})r^2/5]$$

$$-(r^2/5)[{}^4(\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho}) + {}^4(\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})]\} \tag{45}$$

$$\varphi_3^{1,3,k} = (N_0/k)(15)^{1/2}\kappa_r\kappa_\rho^3 j_{1,3}(+)^\tau$$

$$\times\{{}^4(\boldsymbol{\rho})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\mathbf{r})\rho^2/5]$$

$$-(\rho^2/5)[{}^4(\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho}) + {}^4(\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})]\} \tag{46}$$

However, they also give the three difficult functions

$$\varphi_{3s}^{2,2,k} = (N_0/k)(15/2)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau$$

$$\times{}^4(\mathbf{k})[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - r^2\rho^2/3] \tag{47}$$

$$\varphi_{3b}^{2,2,k} = (N_0/k)(15/2)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau$$

$$\times[{}^4(\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) - {}^4(\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})] \tag{48}$$

$$\varphi_{3c}^{2,2,k} = (N_0/k)(135/14)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau$$

$$\times\{{}^4(\mathbf{r})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\mathbf{r})(2\rho^2/3)]$$

$$+{}^4(\boldsymbol{\rho})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\boldsymbol{\rho})(2r^2/3)]$$

$$-(2/3)\,{}^4(\mathbf{k})[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - 2r^2\rho^2/3]\} \tag{49}$$

The difficulty lies in discovering the correct linear combinations to use, which will symmetrize the matrix versions of the operators. In particular, comparison of (47) with (42) discloses a distinct difference in character.

Examples of other functions in this same family are

$$\varphi_3^{4,2,k} = (N_0/k)(525/8)^{1/2}\kappa_r^4\kappa_\rho^2 j_{4,2}(-)^\tau$$
$$\times\{{}^4(\mathbf{r})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})^2 - (\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})(2r^2/7) - (\mathbf{k}\cdot\mathbf{r})(r^2\rho^2/7)]$$
$$-(2/7)\,{}^4(\boldsymbol{\rho})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})r^2 - (\mathbf{k}\cdot\boldsymbol{\rho})r^4/5]$$
$$-(1/7)\,{}^4(\mathbf{k})[(\mathbf{r}\cdot\boldsymbol{\rho})^2 r^2 - r^4\rho^2/5]\} \tag{50}$$

$$\varphi_3^{4,1,2k} = (iN_0/k^2)(525/8)^{1/2}\kappa_r^4\kappa_\rho j_{4,1}(+)^\tau$$
$$\times\{{}^4(\mathbf{r})[(\mathbf{k}\cdot\mathbf{r})^2(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})(2r^2/7) - (\mathbf{r}\cdot\boldsymbol{\rho})(k^2r^2/7)]$$
$$-(2/7)\,{}^4(\mathbf{k})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})r^2 - (\mathbf{k}\cdot\boldsymbol{\rho})r^4/5]$$
$$-(1/7)\,{}^4(\boldsymbol{\rho})[(\mathbf{k}\cdot\mathbf{r})^2 r^2 - k^2r^4/5]\} \tag{51}$$

$$\varphi_3^{1,0,2k} = (iN_0/k^2)(3)^{1/2}\kappa_r j_{1,0}(-)^\tau[{}^4(\mathbf{k})(\mathbf{k}\cdot\mathbf{r}) - {}^4(\mathbf{r})k^2/3] \tag{52}$$

$$\varphi_3^{0,1,2k} = (iN_0/k^2)(3)^{1/2}\kappa_\rho j_{0,1}(+)^\tau[{}^4(\mathbf{k})(\mathbf{k}\cdot\boldsymbol{\rho}) - {}^4(\boldsymbol{\rho})k^2/3] \tag{53}$$

$$\varphi_{3s}^{2,1,2k} = (iN_0/k^2)(27/2)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau$$
$$\times\{{}^4(\mathbf{k})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\boldsymbol{\rho})r^2/3]$$
$$-(1/3)[{}^4(\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})k^2 - {}^4(\boldsymbol{\rho})k^2r^2/3]\} \tag{54}$$

$$\varphi_{3s}^{1,2,2k} = (iN_0/k^2)(27/2)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$
$$\times\{{}^4(\mathbf{k})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\mathbf{r})\rho^2/3]$$
$$-(1/3)[{}^4(\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})k^2 - {}^4(\mathbf{r})k^2\rho^2/3]\} \tag{55}$$

$$\varphi_{3b}^{2,1,2k} = (iN_0/k^2)(10)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau$$
$$\times\{{}^4(\mathbf{r})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho}) - (\mathbf{r}\cdot\boldsymbol{\rho})k^2/2] - {}^4(\boldsymbol{\rho})[(\mathbf{k}\cdot\mathbf{r})^2 - k^2r^2/2]$$
$$+(1/2)\,{}^4(\mathbf{k})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\boldsymbol{\rho})r^2]\} \tag{56}$$

$$\varphi_{3b}^{1,2,2k} = \left(iN_0/k^2\right)(10)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$
$$\times\left\{{}^4(\boldsymbol{\rho})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/2\right]-{}^4(\mathbf{r})\left[(\mathbf{k}\cdot\boldsymbol{\rho})^2-k^2\rho^2/2\right]\right.$$
$$\left.+(1/2)\,{}^4(\mathbf{k})\left[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})\rho^2\right]\right\} \tag{57}$$

$$\varphi_{3c}^{2,1,2k} = \left(iN_0/k^2\right)(50/7)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau$$
$$\times\left\{{}^4(\mathbf{r})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/5\right]+(1/2)\,{}^4(\boldsymbol{\rho})\left[(\mathbf{k}\cdot\mathbf{r})^2-k^2r^2/5\right]\right.$$
$$\left.-(2/5)\,{}^4(\mathbf{k})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})+(\mathbf{k}\cdot\boldsymbol{\rho})r^2/2\right]\right\} \tag{58}$$

$$\varphi_{3c}^{1,2,2k} = \left(iN_0/k^2\right)(50/7)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$
$$\times\left\{{}^4(\boldsymbol{\rho})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/5\right]+(1/2)\,{}^4(\mathbf{r})\left[(\mathbf{k}\cdot\boldsymbol{\rho})^2-k^2\rho^2/5\right]\right.$$
$$\left.-(2/5)\,{}^4(\mathbf{k})\left[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})+(\mathbf{k}\cdot\mathbf{r})\rho^2/2\right]\right\} \tag{59}$$

$$\varphi_{3}^{1,1,3k} = \left(N_0/k^3\right)(15)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^\tau$$
$$\times\left\{{}^4(\mathbf{k})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/5\right]\right.$$
$$\left.-(k^2/5)\left[{}^4(\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})+{}^4(\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})\right]\right\} \tag{60}$$

In addition to the odd-parity ^4P family given above, there is an even-parity ^4P family growing out of the function φ_9 in (38). The first two momentum-dependent functions in this family are the very simple ones:

$$\varphi_9^{1,0,k} = (iN_0/k)\kappa_r j_{1,0}(-)^{\tau\,4}(i\mathbf{k}\times\mathbf{r}) \tag{61}$$

$$\varphi_9^{0,1,k} = (iN_0/k)\kappa_\rho j_{0,1}(+)^{\tau\,4}(i\mathbf{k}\times\boldsymbol{\rho}) \tag{62}$$

Sorting out the next functions proved a difficult task, but eventually led to:

$$\varphi_{9s}^{2,1,k} = (iN_0/k)(9/2)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau\left[{}^4(i\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{k}\times\boldsymbol{\rho})r^2/3\right] \tag{63}$$

$$\varphi_{9a}^{2,1,k} = (iN_0/k)(6)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau$$
$$\times\left[{}^4(i\mathbf{r}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})+(1/2)\,{}^4(i\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{k}\times\boldsymbol{\rho})r^2/2\right] \tag{64}$$

$$\varphi_{9s}^{1,2,k}=(iN_0/k)(9/2)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau\left[{}^4(i\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{k}\times\mathbf{r})\rho^2/3\right] \tag{65}$$

$$\varphi_{9a}^{1,2,k}=(iN_0/k)(6)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$

$$\times\left[{}^4(i\mathbf{r}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho})-(1/2)\,{}^4(i\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})+{}^4(i\mathbf{k}\times\mathbf{r})\rho^2/2\right] \tag{66}$$

Moving in this same direction, the next functions in this family are

$$\varphi_{9s}^{3,2,k}=(iN_0/k)(75/4)^{1/2}\kappa_r^3\kappa_\rho^2 j_{3,2}(-)^\tau$$

$$\times\left\{{}^4(i\mathbf{k}\times\mathbf{r})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^4(i\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})(2r^2/5)\right\} \tag{67}$$

$$\varphi_{9a}^{3,2,k}=(iN_0/k)(75/2)^{1/2}\kappa_r^3\kappa_\rho^2 j_{3,2}(-)^\tau$$

$$\times\left\{{}^4(i\mathbf{r}\times\boldsymbol{\rho})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})r^2/5\right]\right.$$

$$\left.+(1/2)\,{}^4(i\mathbf{k}\times\mathbf{r})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^4(i\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})(2r^2/5)\right\} \tag{68}$$

$$\varphi_{9s}^{2,3,k}=(iN_0/k)(75/4)^{1/2}\kappa_r^2\kappa_\rho^3 j_{2,3}(+)^\tau$$

$$\times\left\{{}^4(i\mathbf{k}\times\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^4(i\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})(2\rho^2/5)\right\} \tag{69}$$

$$\varphi_{9a}^{2,3,k}=(iN_0/k)(75/2)^{1/2}\kappa_r^2\kappa_\rho^3 j_{2,3}(+)^\tau$$

$$\times\left\{{}^4(i\mathbf{r}\times\boldsymbol{\rho})\left[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})\rho^2/5\right]\right.$$

$$\left.-(1/2)\,{}^4(i\mathbf{k}\times\boldsymbol{\rho})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]+{}^4(i\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})(2\rho^2/5)\right\} \tag{70}$$

Functions with quadratic dependence upon momentum include the very simple pair

$$\varphi_9^{2,0,2k}=(N_0/k^2)(5)^{1/2}\kappa_r^2 j_{2,0}(-)^{\tau}\,{}^4(i\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\mathbf{r}) \tag{71}$$

$$\varphi_9^{0,2,2k}=(N_0/k^2)(5)^{1/2}\kappa_\rho^2 j_{0,2}(-)^{\tau}\,{}^4(i\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho}) \tag{72}$$

and the somewhat more complicated linear combinations:

$$\varphi_{9s}^{1,1,2k}=\left(N_0/k^2\right)(3/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^\tau\left[{}^4(i\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})+{}^4(i\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})\right] \tag{73}$$

$$\varphi_{9a}^{1,1,2k}=\left(N_0/k^2\right)(9/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^\tau$$
$$\times\left[{}^4(i\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})+{}^4(i\mathbf{r}\times\boldsymbol{\rho})(2k^2/3)\right] \tag{74}$$

The system of functions is, of course, endless. However, we will actually need to use only a relatively small number of these functions, at least in the calculations needed for the identification of particlelike trilocal structures.

When the auxiliary operator equations (8)–(10) are applied to the part of the wave function containing these ${}^4P$ functions, we are able to solve for the coefficients of these functions, in terms of a few leading coefficients. These leading coefficients will include C_3, C_4, and C_9, the coefficients belonging to the functions (36)–(38). Also, at this stage, they will include $C_3^{0,0,k}$, $C_9^{1,0,k}$, and $C_9^{0,1,k}$, belonging to (39) and (61), (62), but at a later stage we will express the latter three in terms of the former three, with the aid of (11).

For the functions (40)–(44), the coefficients are

$$C_3^{2,0,k}=(9/2)^{1/2}\left[\nu'C_3+(1/3)C_3^{0,0,k}\right] \tag{75}$$

$$C_3^{0,2,k}=(9/2)^{1/2}\left[\nu''C_3+(1/3)C_3^{0,0,k}\right] \tag{76}$$

$$C_{3s}^{1,1,k}=(3/5)^{1/2}\left[\nu''C_3+\nu'C_4-\nu C_3^{0,0,k}\right] \tag{77}$$

$$C_{3b}^{1,1,k}=(3/2)[\nu''C_3-\nu'C_4] \tag{78}$$

$$C_{3c}^{1,1,k}=(3/4)^{1/2}\left[\nu''C_3+\nu'C_4+2\nu C_3^{0,0,k}\right] \tag{79}$$

The coefficients for the functions (45)–(49) are found to be

$$C_3^{3,1,k}=(45/2)^{1/2}\left[(-\nu\nu'+\nu''/5)C_3+(\nu'/5)C_4-(\nu/5)C_3^{0,0,k}\right] \tag{80}$$

$$C_3^{1,3,k}=(45/2)^{1/2}\left[(-\nu\nu''+\nu'/5)C_4+(\nu''/5)C_3-(\nu/5)C_3^{0,0,k}\right] \tag{81}$$

$$C_{3s}^{2,2,k}=(45/4)^{1/2}(\nu^2-1/3)C_3^{0,0,k} \tag{82}$$

$$C_{3b}^{2,2,k}=(45/4)^{1/2}[\nu''C_3-\nu'C_4] \tag{83}$$

$$C_{3c}^{2,2,k}=(405/28)^{1/2}\left[(-\nu\nu''+2\nu'/3)C_3+(-\nu\nu'+2\nu''/3)C_4\right.$$
$$\left.-(2/3)(\nu^2-1/3)C_3^{0,0,k}\right] \tag{84}$$

In the even-parity family, we find that the coefficients to be associated with the functions (63)–(74) have the following reduction formulas:

$$C_{9s}^{2,1,k}=(9/2)^{1/2}\left[-\nu C_9^{1,0,k}+(1/3)C_9^{0,1,k}\right] \tag{85}$$

$$C_{9a}^{2,1,k}=(6)^{1/2}\left[-\nu' C_9-(\nu/2)C_9^{1,0,k}+(1/2)C_9^{0,1,k}\right] \tag{86}$$

$$C_{9s}^{1,2,k}=(9/2)^{1/2}\left[-\nu C_9^{0,1,k}+(1/3)C_9^{1,0,k}\right] \tag{87}$$

$$C_{9a}^{1,2,k}=(6)^{1/2}\left[-\nu'' C_9+(\nu/2)C_9^{0,1,k}-(1/2)C_9^{1,0,k}\right] \tag{88}$$

$$C_{9s}^{3,2,k}=(75/4)^{1/2}\left[(\nu^2-1/5)C_9^{1,0,k}-(2\nu/5)C_9^{0,1,k}\right] \tag{89}$$

$$C_{9a}^{3,2,k}=(75/2)^{1/2}\left[(\nu\nu'-\nu''/5)C_9+(1/2)(\nu^2-1/5)C_9^{1,0,k}\right.$$
$$\left.-(2\nu/5)C_9^{0,1,k}\right] \tag{90}$$

$$C_{9s}^{2,3,k}=(75/4)^{1/2}\left[(\nu^2-1/5)C_9^{0,1,k}-(2\nu/5)C_9^{1,0,k}\right] \tag{91}$$

$$C_{9a}^{2,3,k}=(75/2)^{1/2}\left[(\nu\nu''-\nu'/5)C_9-(1/2)(\nu^2-1/5)C_9^{0,1,k}\right.$$
$$\left.+(2\nu/5)C_9^{1,0,k}\right] \tag{92}$$

$$C_9^{2,0,2k}=(5)^{1/2}\nu' C_9^{1,0,k} \tag{93}$$

$$C_9^{0,2,2k}=(5)^{1/2}\nu'' C_9^{0,1,k} \tag{94}$$

$$C_{9s}^{1,1,2k}=(3/2)^{1/2}\left[\nu'' C_9^{1,0,k}+\nu' C_9^{0,1,k}\right] \tag{95}$$

$$C_{9a}^{1,1,2k}=(9/2)^{1/2}\left[\nu'' C_9^{1,0,k}-\nu' C_9^{0,1,k}+(2/3)C_9\right] \tag{96}$$

We can see that the formulas for the coefficients (85)–(96) are transliterations of the formulas for the functions (63)–(74), in which we have made the replacements

$$^4(i\mathbf{r}\times\boldsymbol{\rho})\to C_9, \qquad ^4(i\mathbf{k}\times\mathbf{r})\to -iC_9^{1,0,k}, \qquad ^4(i\mathbf{k}\times\boldsymbol{\rho})\to -iC_9^{0,1,k} \tag{97}$$

$$(\mathbf{r}\cdot\boldsymbol{\rho})\to-\nu, \qquad (\mathbf{k}\cdot\mathbf{r})\to i\nu', \qquad (\mathbf{k}\cdot\boldsymbol{\rho})\to i\nu'' \tag{98}$$

$$k^2\to+1, \qquad r^2\to-1, \qquad \rho^2\to-1 \tag{99}$$

Other quantities in the functional formulas are replaced by unity, except that the numerical coefficients, the numerical normalization factors, and the factor i when it appears, are left unchanged. In a similar way, the coefficients in (75)–(84) are transliterations of the functions in (40)–(49), using (98) and (99) and the further replacements:

$$
\begin{aligned}
{}^4(\mathbf{k}) &\to (3/2)^{1/2} C_3^{0,0,k} \\
{}^4(\mathbf{r}) &\to -i(3/2)^{1/2} C_3 \\
{}^4(\boldsymbol{\rho}) &\to -i(3/2)^{1/2} C_4
\end{aligned} \tag{100}
$$

The formulas for the coefficients were obtained from the operator equations (8)–(10). We still have (11) to make use of. This couples together the odd-parity and even-parity functions. Operations upon the leading odd-parity ${}^4P$ functions lead to the results

$$
\begin{aligned}
(P^\tau/k\kappa_r\kappa_\rho)(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\varphi_3 = {} & (2/27)^{1/2}\varphi_9^{0,1,k} \\
& + (1/27)^{1/2}\varphi_{9s}^{2,1,k} + (1/3)\varphi_{9a}^{2,1,k}
\end{aligned} \tag{101}
$$

$$
\begin{aligned}
(P^\tau/k\kappa_r\kappa_\rho)(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\varphi_4 = {} & -(2/27)^{1/2}\varphi_9^{1,0,k} \\
& - (1/27)^{1/2}\varphi_{9s}^{1,2,k} - (1/3)\varphi_{9a}^{1,2,k}
\end{aligned} \tag{102}
$$

$$
(P^\tau/k\kappa_r\kappa_\rho)(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\varphi_3^{0,0,k} = (2/27)^{1/2}\varphi_9 + (4/27)^{1/2}\varphi_{9a}^{1,1,2k} \tag{103}
$$

Similar operations upon the leading even-parity ${}^4P$ functions give

$$
\begin{aligned}
(P^\tau/k\kappa_r\kappa_\rho)(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\varphi_9 = {} & -(2/27)^{1/2}\varphi_3^{0,0,k} \\
& - (1/27)^{1/2}\varphi_3^{2,0,k} - (1/27)^{1/2}\varphi_3^{0,2,k} \\
& + (2/27)^{1/2}\varphi_{3s}^{2,2,k} - (14/135)^{1/2}\varphi_{3c}^{2,2,k}
\end{aligned} \tag{104}
$$

$$\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_9^{1,0,k}=(2/27)^{1/2}\varphi_4$$
$$+(1/27)^{1/2}\varphi_{19}-(1/27)^{1/2}\varphi_3^{0,1,2k}$$
$$-(1/54)^{1/2}\varphi_{3s}^{2,1,2k}-(1/10)^{1/2}\varphi_{3b}^{2,1,2k} \tag{105}$$

$$\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_9^{0,1,k}=-(2/27)^{1/2}\varphi_3$$
$$-(1/27)^{1/2}\varphi_{20}+(1/27)^{1/2}\varphi_3^{1,0,2k}$$
$$+(1/54)^{1/2}\varphi_{3s}^{1,2,2k}+(1/10)^{1/2}\varphi_{3b}^{1,2,2k} \tag{106}$$

The functions φ_{19} and φ_{20}, which appear above, are defined in Appendix A of III, and the coefficients C_{19} and C_{20}, which will be needed in what follows, were given in Appendix E of III. We will also need the following four coefficients:

$$C_{3s}^{2,1,2k}=(9/2)\left[(\nu\nu'-\nu''/3)C_3^{0,0,k}+(\nu/3)C_3-(1/9)C_4\right] \tag{107}$$

$$C_{3s}^{1,2,2k}=(9/2)\left[(\nu\nu''-\nu'/3)C_3^{0,0,k}+(\nu/3)C_4-(1/9)C_3\right] \tag{108}$$

$$C_{3b}^{2,1,2k}=(15)^{1/2}\left[(-\nu'\nu''+\nu/2)C_3+(\nu'^2-1/2)C_4\right.$$
$$\left.+(1/2)(\nu\nu'-\nu'')C_3^{0,0,k}\right] \tag{109}$$

$$C_{3b}^{1,2,2k}=(15)^{1/2}\left[(-\nu'\nu''+\nu/2)C_4+(\nu''^2-1/2)C_3\right.$$
$$\left.+(1/2)(\nu\nu''-\nu')C_3^{0,0,k}\right] \tag{110}$$

As mentioned earlier, the operator in (101)–(106) is skew symmetric. Inspection of the first matrix element on the right of each of these six equations will provide illustrations of the sign changes. When the operator relationships are replaced by relations among coefficients, this skew symmetry needs to be allowed for. It is most convenient to put the sign change on the left of the equals sign, giving six relations, of which the first is

$$\gamma C_3=(2/27)^{1/2}C_9^{0,1,k}+(1/27)^{1/2}C_{9s}^{2,1,k}+(1/3)C_{9a}^{2,1,k}$$
$$=(2/3)^{1/2}\left[-\nu'C_9-\nu C_9^{1,0,k}+C_9^{0,1,k}\right] \tag{111}$$

These six equations, after substitution from the appropriate reduction equations, take the form of six linear homogeneous equations in six unknowns. Because of redundancies, it is not possible to solve for five unknowns in terms of the sixth, but it is possible to solve for three in terms of the other three. There is a secular equation that needs to be satisfied, but it is identical with (13) and therefore has already been satisfied as a result of algebraic relationships that hold among the operators that appear in (8)–(11).

The three new reduction equations are

$$C_3^{0,0,k} = (1-\nu^2)^{-1}\left[(\nu\nu''-\nu')C_3 + (\nu\nu'-\nu'')C_4 - (2/3)^{1/2}\gamma C_9\right] \quad (112)$$

$$C_9^{1,0,k} = (1-\nu^2)^{-1}\left[(3/2)^{1/2}\gamma(\nu C_3 - C_4) + (\nu\nu'-\nu'')C_9\right] \quad (113)$$

$$C_9^{0,1,k} = (1-\nu^2)^{-1}\left[(3/2)^{1/2}\gamma(C_3 - \nu C_4) - (\nu\nu''-\nu')C_9\right] \quad (114)$$

With these three now available, all of the higher ^4P momentum-dependent coefficients can be reduced to linear combinations of the three rest-system coefficients C_3, C_4, and C_9. The parameters ν, ν', ν'', and γ will all appear, of course, and magnitudes for these parameters will need to be determined as a part of a trilocal solution, but the infinite matrix equations that would otherwise need to be solved are by this procedure reduced to finite matrix equations. For the ^4P families, only these three expansion coefficients remain, at this stage, independently adjustable.

6. ^2P-STATE FAMILIES

The doublet P-state families can be set up in direct analogy with the quartet P-state families in the previous section. Note first the rest-system functions having the following definitions:

$$\varphi_5 = iN_0\kappa_r j_{1,0}\left[(+)^{\tau 2b}(\mathbf{r}) + (-)^{\tau 2c}(\mathbf{r})\right] \quad (115)$$

$$\varphi_7 = iN_0\kappa_\rho j_{0,1}\left[(-)^{\tau 2b}(\boldsymbol{\rho}) + (+)^{\tau 2c}(\boldsymbol{\rho})\right] \quad (116)$$

$$\varphi_{13} = N_0(3/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}\left[(-)^{\tau 2b}(i\mathbf{r}\times\boldsymbol{\rho}) + (+)^{\tau 2c}(i\mathbf{r}\times\boldsymbol{\rho})\right] \quad (117)$$

Introduce now a momentum-dependent trio of functions:

$$\varphi_5^{0,0,k} = (N_0/k)j_{0,0}\left[(+)^{\tau 2b}(\mathbf{k}) + (-)^{\tau 2c}(\mathbf{k})\right] \quad (118)$$

$$\varphi_{13}^{1,0,k} = (iN_0/k)(3/2)^{1/2}\kappa_r j_{1,0}\left[(+)^{\tau 2b}(i\mathbf{k}\times\mathbf{r}) + (-)^{\tau 2c}(i\mathbf{k}\times\mathbf{r})\right] \quad (119)$$

$$\varphi_{13}^{0,1,k} = (iN_0/k)(3/2)^{1/2}\kappa_\rho j_{0,1}\left[(-)^{\tau 2b}(i\mathbf{k}\times\boldsymbol{\rho}) + (+)^{\tau 2c}(i\mathbf{k}\times\boldsymbol{\rho})\right] \quad (120)$$

The substitutions which will change the ^4P functions in (36)–(39), (61), and (62) into the ^2P functions in (115)–(120) can now be used to change any of the ^4P functions in the previous section into analogous ^2P functions in the new family.

When the analogy is followed further, we find that the coefficients of these families of ^2P functions can be expressed as multiples of the three leading coefficients, C_5, C_7, and C_{13}.

In addition to the above pair of families, there is a parallel system which starts from the rest-system functions:

$$\varphi_6 = iN_0(1/3)^{1/2}\kappa_r j_{1,0}\left[3(+)^{\tau 2b}(\mathbf{r}) - (-)^{\tau 2c}(\mathbf{r})\right] \tag{121}$$

$$\varphi_8 = iN_0(1/3)^{1/2}\kappa_\rho j_{0,1}\left[3(-)^{\tau 2b}(\boldsymbol{\rho}) - (+)^{\tau 2c}(\boldsymbol{\rho})\right] \tag{122}$$

$$\varphi_{14} = N_0(1/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}\left[3(-)^{\tau 2b}(i\mathbf{r}\times\boldsymbol{\rho}) - (+)^{\tau 2c}(i\mathbf{r}\times\boldsymbol{\rho})\right] \tag{123}$$

The leading momentum-dependent functions in this system are

$$\varphi_6^{0,0,k} = (N_0/k)(1/3)^{1/2} j_{0,0}\left[3(+)^{\tau 2b}(\mathbf{k}) - (-)^{\tau 2c}(\mathbf{k})\right] \tag{124}$$

$$\varphi_{14}^{1,0,k} = (iN_0/k)(1/2)^{1/2}\kappa_r j_{1,0}\left[3(+)^{\tau 2b}(i\mathbf{k}\times\mathbf{r}) - (-)^{\tau 2c}(i\mathbf{k}\times\mathbf{r})\right] \tag{125}$$

$$\varphi_{14}^{0,1,k} = (iN_0/k)(1/2)^{1/2}\kappa_\rho j_{0,1}\left[3(-)^{\tau 2b}(i\mathbf{k}\times\boldsymbol{\rho}) - (+)^{\tau 2c}(i\mathbf{k}\times\boldsymbol{\rho})\right] \tag{126}$$

Again, with appropriate substitutions each of the ^4P functions in the previous section can be used for the generation of a corresponding function in the above system of ^2P functions. For these families, the coefficients can be expressed as multiples of the three leading coefficients, C_6, C_8, and C_{14}.

There is therefore no need to repeat the algebra for these ^2P families, based on the functions in (115)–(126). It is the same algebra as used for the ^4P families in the previous section.

The function translations which will take us from φ_3 to φ_5, from φ_4 to φ_7, and from $\varphi_3^{0,0,k}$ to $\varphi_5^{0,0,k}$ can be summarized as

$$(\pm)^{\tau 4}(\ ) \to (3/2)^{1/2}\left[(\mp)^{\tau 2b}(\ ) + (\pm)^{\tau 2c}(\ )\right] \tag{127}$$

where the vector in the open parentheses is whatever vector is involved in the translation. This same translation (127) can also be used with the even-parity functions, and will take us from φ_9 to φ_{13}, from $\varphi_9^{1,0,k}$ to $\varphi_{13}^{1,0,k}$, and from $\varphi_9^{0,1,k}$ to $\varphi_{13}^{0,1,k}$.

Each of the above six translations can be considered as a syllable translation within more complicated word translations that are involved when functions such as (56) or (70) are converted from ^4P functions to ^2P functions in these families based on (115)–(120).

Functions in the ^2P families based on (121)–(126) can be obtained from ^4P functions with translations summarized as

$$(\pm)^{\tau 4}(\;)\to(1/2)^{1/2}\left[3(\mp)^{\tau 2b}(\;)-(\pm)^{\tau 2c}(\;)\right] \tag{128}$$

which carry φ_3 to φ_6, φ_4 to φ_8, φ_9 to φ_{14}, and so forth.

The coefficients for the ^2P functions can be obtained as transliterations of the functions themselves, by a procedure parallel to that used for the ^4P families, except that instead of the replacements (100) the following replacements are needed:

$$\begin{aligned}
&(\mp)^{\tau 2b}(\mathbf{k})+(\pm)^{\tau 2c}(\mathbf{k})\to C_5^{0,0,k}\\
&(\mp)^{\tau 2b}(\mathbf{r})+(\pm)^{\tau 2c}(\mathbf{r})\to -iC_5\\
&(\mp)^{\tau 2b}(\boldsymbol{\rho})+(\pm)^{\tau 2c}(\boldsymbol{\rho})\to -iC_7
\end{aligned} \tag{129}$$

$$\begin{aligned}
&3(\mp)^{\tau 2b}(\mathbf{k})-(\pm)^{\tau 2c}(\mathbf{k})\to(3)^{1/2}C_6^{0,0,k}\\
&3(\mp)^{\tau 2b}(\mathbf{r})-(\pm)^{\tau 2c}(\mathbf{r})\to -i(3)^{1/2}C_6\\
&3(\mp)^{\tau 2b}(\boldsymbol{\rho})-(\pm)^{\tau 2c}(\boldsymbol{\rho})\to -i(3)^{1/2}C_8
\end{aligned} \tag{130}$$

Similarly, instead of (97) we will need to use

$$\begin{aligned}
&(\mp)^{\tau 2b}(i\mathbf{r}\times\boldsymbol{\rho})+(\pm)^{\tau 2c}(i\mathbf{r}\times\boldsymbol{\rho})\to(2/3)^{1/2}C_{13}\\
&(\mp)^{\tau 2b}(i\mathbf{k}\times\mathbf{r})+(\pm)^{\tau 2c}(i\mathbf{k}\times\mathbf{r})\to -i(2/3)^{1/2}C_{13}^{1,0,k}\\
&(\mp)^{\tau 2b}(i\mathbf{k}\times\boldsymbol{\rho})+(\pm)^{\tau 2c}(i\mathbf{k}\times\boldsymbol{\rho})\to -i(2/3)^{1/2}C_{13}^{1,0,k}
\end{aligned} \tag{131}$$

$$\begin{aligned}
&3(\mp)^{\tau 2b}(i\mathbf{r}\times\boldsymbol{\rho})-(\pm)^{\tau 2c}(i\mathbf{r}\times\boldsymbol{\rho})\to(2)^{1/2}C_{14}\\
&3(\mp)^{\tau 2b}(i\mathbf{k}\times\mathbf{r})-(\pm)^{\tau 2c}(i\mathbf{k}\times\mathbf{r})\to -i(2)^{1/2}C_{14}^{1,0,k}\\
&3(\mp)^{\tau 2b}(i\mathbf{k}\times\boldsymbol{\rho})-(\pm)^{\tau 2c}(i\mathbf{k}\times\boldsymbol{\rho})\to -i(2)^{1/2}C_{14}^{0,1,k}
\end{aligned} \tag{132}$$

When the operator in (11) is applied to these ^2P functions, relationships similar to those with the ^4P functions are obtained. In particular, we find that the reduction equations (112)–(114) have doublet analogs, given by

$$C_5^{0,0,k} = (1-\nu^2)^{-1}\left[(\nu\nu''-\nu')C_5 + (\nu\nu'-\nu'')C_7 - (2/3)^{1/2}\gamma C_{13}\right] \tag{133}$$

$$C_{13}^{1,0,k} = (1-\nu^2)^{-1}\left[(3/2)^{1/2}\gamma(\nu C_5 - C_7) + (\nu\nu'-\nu'')C_{13}\right] \tag{134}$$

$$C_{13}^{0,1,k} = (1-\nu^2)^{-1}\left[(3/2)^{1/2}\gamma(C_5 - \nu C_7) - (\nu\nu''-\nu')C_{13}\right] \tag{135}$$

$$C_6^{0,0,k} = (1-\nu^2)^{-1}\left[(\nu\nu''-\nu')C_6 + (\nu\nu'-\nu'')C_8 - (2/3)^{1/2}\gamma C_{14}\right] \tag{136}$$

$$C_{14}^{1,0,k} = (1-\nu^2)^{-1}\left[(3/2)^{1/2}\gamma(\nu C_6 - C_8) + (\nu\nu'-\nu'')C_{14}\right] \tag{137}$$

$$C_{14}^{0,1,k} = (1-\nu^2)^{-1}\left[(3/2)^{1/2}\gamma(C_6 - \nu C_8) - (\nu\nu''-\nu')C_{14}\right] \tag{138}$$

With the use of these reduction equations, all of the ${}^2P$ coefficients can be reduced to linear combinations of six head-of-family coefficients. These six are C_5, C_6, C_7, C_8, C_{13}, and C_{14}. In the linear combinations, there is involvement of the eigenvalue parameters ν, ν', ν'', and γ. This involvement is very similar to the involvement in the reduction of the ${}^2S$ and ${}^4P$ coefficients. It is a part of the program by which the infinite expansion will be reduced to a finite expansion in terms of 16 unknown head-of-family coefficients.

7. ${}^4D$-STATE FAMILIES

There are five quartet D-state functions within the first 16 rest-system tree functions. Three of these have even parity:

$$\varphi_{10} = N_0(6)^{1/2}\kappa_r^2 j_{2,0}(-)^{\tau 4}(\mathbf{rr}) \tag{139}$$

$$\varphi_{11} = N_0(6)^{1/2}\kappa_\rho^2 j_{0,2}(-)^{\tau 4}(\boldsymbol{\rho\rho}) \tag{140}$$

$$\varphi_{12} = N_0(9/5)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^{\tau 4}(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r}) \tag{141}$$

The other two have odd parity:

$$\varphi_{15} = iN_0(12)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^{\tau 4}(i\mathbf{rr}\times\boldsymbol{\rho}) \tag{142}$$

$$\varphi_{16} = iN_0(12)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^{\tau 4}(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho}) \tag{143}$$

Operations upon (139)–(141), using the operators in (8)–(10), generate a ${}^4D$ family which includes many momentum-dependent tree functions. Some of them contain the grouping in (7), which vanishes identically, so that these are to be treated as null functions. This ${}^4D$ family has even parity. Similar operations upon (142) and (143) generate an odd-parity ${}^4D$ family, which includes a few functions containing the grouping in (6). These are also to be treated as null functions.

The even-parity family contains the three leading momentum-dependent functions:

$$\varphi_{10}^{1,0,k} = (iN_0/k)(9/5)^{1/2}\kappa_r j_{1,0}(-)^{\tau\,4}(\mathbf{kr}+\mathbf{rk}) \tag{144}$$

$$\varphi_{10}^{0,1,k} = (iN_0/k)(9/5)^{1/2}\kappa_\rho j_{0,1}(+)^{\tau\,4}(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k}) \tag{145}$$

$$\varphi_{10}^{0,0,2k} = (N_0/k^2)(6)^{1/2} j_{0,0}(-)^{\tau\,4}(\mathbf{kk}) \tag{146}$$

Functions linear in the momentum vector include the following:

$$\begin{aligned}\varphi_{10s}^{2,1,k} &= (iN_0/k)(81/10)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau \\ &\quad\times\left[{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})r^2/3\right]\end{aligned} \tag{147}$$

$$\begin{aligned}\varphi_{10s}^{1,2,k} &= (iN_0/k)(81/10)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau \\ &\quad\times\left[{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\mathbf{kr}+\mathbf{rk})\rho^2/3\right]\end{aligned} \tag{148}$$

$$\begin{aligned}\varphi_{10b}^{2,1,k} &= (iN_0/k)(24)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau \\ &\quad\times\left[{}^4(\mathbf{rr})(\mathbf{k}\cdot\boldsymbol{\rho})-(1/2)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\mathbf{r})\right. \\ &\quad\left.-(1/4)\,{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{r}\cdot\boldsymbol{\rho})+{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})r^2/4\right]\end{aligned} \tag{149}$$

$$\begin{aligned}\varphi_{10b}^{1,2,k} &= (iN_0/k)(24)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau \\ &\quad\times\left[{}^4(\boldsymbol{\rho}\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})-(1/2)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})\right. \\ &\quad\left.-(1/4)\,{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})+{}^4(\mathbf{kr}+\mathbf{rk})\rho^2/4\right]\end{aligned} \tag{150}$$

$$\begin{aligned}\varphi_{10c}^{2,1,k} &= (iN_0/k)(30/7)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau \\ &\quad\times\left[{}^4(\mathbf{rr})(\mathbf{k}\cdot\boldsymbol{\rho})+{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\mathbf{r})\right. \\ &\quad\left.-(2/5)\,{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})r^2/5\right]\end{aligned} \tag{151}$$

$$\varphi_{10c}^{1,2,k}=(iN_0/k)(30/7)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$
$$\times\left[{}^4(\boldsymbol{\rho}\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})+{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})\right.$$
$$\left.-(2/5)\,{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(\mathbf{kr}+\mathbf{rk})\rho^2/5\right] \tag{152}$$

$$\varphi_{10}^{3,0,k}=(iN_0/k)(30)^{1/2}\kappa_r^3 j_{3,0}(-)^\tau\left[{}^4(\mathbf{rr})(\mathbf{k}\cdot\mathbf{r})-{}^4(\mathbf{kr}+\mathbf{rk})r^2/5\right] \tag{153}$$

$$\varphi_{10}^{0,3,k}=(iN_0/k)(30)^{1/2}\kappa_\rho^3 j_{0,3}(+)^\tau\left[{}^4(\boldsymbol{\rho}\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho})-{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\rho^2/5\right] \tag{154}$$

$$\varphi_{10s}^{3,2,k}=(iN_0/k)(135/4)^{1/2}\kappa_r^3\kappa_\rho^2 j_{3,2}(-)^\tau$$
$$\times\left\{{}^4(\mathbf{kr}+\mathbf{rk})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]-{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})(2r^2/5)\right\} \tag{155}$$

$$\varphi_{10s}^{2,3,k}=(iN_0/k)(135/4)^{1/2}\kappa_r^2\kappa_\rho^3 j_{2,3}(+)^\tau$$
$$\times\left\{{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]\right.$$
$$\left.-{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{r}\cdot\boldsymbol{\rho})(2\rho^2/5)\right\} \tag{156}$$

$$\varphi_{10b}^{3,2,k}=(iN_0/k)(150)^{1/2}\kappa_r^3\kappa_\rho^2 j_{3,2}(-)^\tau$$
$$\times\left\{{}^4(\mathbf{rr})(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(1/2)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})+(\mathbf{k}\cdot\boldsymbol{\rho})r^2/5\right]\right.$$
$$+{}^4(\boldsymbol{\rho}\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})r^2/5-(1/4)\,{}^4(\mathbf{kr}+\mathbf{rk})$$
$$\left.\times\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]+{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})r^2/5\right\} \tag{157}$$

$$\varphi_{10b}^{2,3,k}=(iN_0/k)(150)^{1/2}\kappa_r^2\kappa_\rho^3 j_{2,3}(+)^\tau$$
$$\times\left\{{}^4(\boldsymbol{\rho}\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(1/2)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\left[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})+(\mathbf{k}\cdot\mathbf{r})\rho^2/5\right]\right.$$
$$+{}^4(\mathbf{rr})(\mathbf{k}\cdot\boldsymbol{\rho})\rho^2/5-(1/4)\,{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})$$
$$\left.\times\left[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5\right]+{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{r}\cdot\boldsymbol{\rho})\rho^2/5\right\} \tag{158}$$

$$\varphi_{10c}^{3,2,k}=(iN_0/k)(125/2)^{1/2}\kappa_r^3\kappa_\rho^2 j_{3,2}(-)^\tau$$
$$\times\{^4(\mathbf{rr})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})\rho^2]$$
$$+^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})(2r^2/5)]$$
$$-^4(\boldsymbol{\rho\rho})(\mathbf{k}\cdot\mathbf{r})(2r^2/5)-(2/5)\,^4(\mathbf{kr}+\mathbf{rk})$$
$$\times[(\mathbf{r}\cdot\boldsymbol{\rho})^2-7r^2\rho^2/10]-^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})r^2/25\} \tag{159}$$

$$\varphi_{10c}^{2,3,k}=(iN_0/k)(125/2)^{1/2}\kappa_r^2\kappa_\rho^3 j_{2,3}(+)^\tau$$
$$\times\{^4(\boldsymbol{\rho\rho})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})r^2]$$
$$+^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})(2\rho^2/5)]$$
$$-^4(\mathbf{rr})(\mathbf{k}\cdot\boldsymbol{\rho})(2\rho^2/5)-(2/5)\,^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})$$
$$\times[(\mathbf{r}\cdot\boldsymbol{\rho})^2-7r^2\rho^2/10]-^4(\mathbf{kr}+\mathbf{rk})(\mathbf{r}\cdot\boldsymbol{\rho})\rho^2/25\} \tag{160}$$

$$\varphi_{10}^{4,1,k}=(iN_0/k)(315/2)^{1/2}\kappa_r^4\kappa_\rho j_{4,1}(+)^\tau$$
$$\times\{^4(\mathbf{rr})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})r^2/7]-^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\mathbf{r})r^2/7$$
$$-^4(\mathbf{kr}+\mathbf{rk})(\mathbf{r}\cdot\boldsymbol{\rho})r^2/7+^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})r^4/35\} \tag{161}$$

$$\varphi_{10}^{1,4,k}=(iN_0/k)(315/2)^{1/2}\kappa_r\kappa_\rho^4 j_{1,4}(-)^\tau$$
$$\times\{^4(\boldsymbol{\rho\rho})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})\rho^2/7]-^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})\rho^2/7$$
$$-^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})\rho^2/7+^4(\mathbf{kr}+\mathbf{rk})\rho^4/35\} \tag{162}$$

Functions quadratic in the momentum vector include

$$\varphi_{10}^{2,0,2k}=(N_0/k^2)(135/7)^{1/2}\kappa_r^2 j_{2,0}(-)^\tau$$
$$\times[^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\mathbf{r})-^4(\mathbf{rr})(2k^2/3)-^4(\mathbf{kk})(2r^2/3)] \tag{163}$$

$$\varphi_{10}^{0,2,2k}=(N_0/k^2)(135/7)^{1/2}\kappa_\rho^2 j_{0,2}(-)^\tau$$
$$\times[^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\boldsymbol{\rho})-^4(\boldsymbol{\rho\rho})(2k^2/3)-^4(\mathbf{kk})(2\rho^2/3)] \tag{164}$$

$$\varphi_{10s}^{1,1,2k}=\left(N_0/k^2\right)(18)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^{\tau\,4}(\mathbf{kk})(\mathbf{r}\cdot\boldsymbol{\rho}) \tag{165}$$

$$\varphi_{10b}^{1,1,2k}=\left(N_0/k^2\right)(9/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^{\tau}\times\left[{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\boldsymbol{\rho})-{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\mathbf{r})\right] \tag{166}$$

$$\varphi_{10c}^{1,1,2k}=\left(N_0/k^2\right)(81/14)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^{\tau}\times\left[{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\boldsymbol{\rho})+{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\mathbf{r})-(2/3)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})k^2-(4/3)\,{}^4(\mathbf{kk})(\mathbf{r}\cdot\boldsymbol{\rho})\right] \tag{167}$$

$$\varphi_{10}^{4,0,2k}=\left(N_0/k^2\right)(525/4)^{1/2}\kappa_r^4 j_{4,0}(-)^{\tau}\times\left\{{}^4(\mathbf{rr})\left[(\mathbf{k}\cdot\mathbf{r})^2-k^2r^2/7\right]-{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\mathbf{r})\left(2r^2/7\right)+{}^4(\mathbf{kk})\left(2r^4/35\right)\right\} \tag{168}$$

$$\varphi_{10}^{0,4,2k}=\left(N_0/k^2\right)(525/4)^{1/2}\kappa_\rho^4 j_{0,4}(-)^{\tau}\times\left\{{}^4(\boldsymbol{\rho\rho})\left[(\mathbf{k}\cdot\boldsymbol{\rho})^2-k^2\rho^2/7\right]-{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\boldsymbol{\rho})\left(2\rho^2/7\right)+{}^4(\mathbf{kk})\left(2\rho^4/35\right)\right\} \tag{169}$$

$$\varphi_{10s}^{3,1,2k}=\left(N_0/k^2\right)(675/7)^{1/2}\kappa_r^3\kappa_\rho j_{3,1}(+)^{\tau}\times\left\{{}^4(\mathbf{kr}+\mathbf{rk})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})r^2/5\right]-{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\mathbf{r})r^2/5-{}^4(\mathbf{kk})(\mathbf{r}\cdot\boldsymbol{\rho})\left(2r^2/5\right)-(2/3)\left[{}^4(\mathbf{rr})(\mathbf{r}\cdot\boldsymbol{\rho})k^2-{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})k^2r^2/5\right]\right\} \tag{170}$$

$$\varphi_{10s}^{1,3,2k}=\left(N_0/k^2\right)(675/7)^{1/2}\kappa_r\kappa_\rho^3 j_{1,3}(+)^{\tau}\times\left\{{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\left[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})\rho^2/5\right]-{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\boldsymbol{\rho})\rho^2/5-{}^4(\mathbf{kk})(\mathbf{r}\cdot\boldsymbol{\rho})\left(2\rho^2/5\right)-(2/3)\left[{}^4(\boldsymbol{\rho\rho})(\mathbf{r}\cdot\boldsymbol{\rho})k^2-{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})k^2\rho^2/5\right]\right\} \tag{171}$$

$$\begin{aligned}\varphi_{10b}^{3,1,2k} = {} & \big(N_0/k^2\big)(675/4)^{1/2}\kappa_r^3\kappa_\rho j_{3,1}(+)^\tau \\ & \times\big\{{}^4(\mathbf{rr})\big[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/3\big] \\ & -(1/2)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\big[(\mathbf{k}\cdot\mathbf{r})^2-k^2r^2/3\big] \\ & -{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\boldsymbol{\rho})r^2/5+{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\mathbf{r})r^2/5\big\}\end{aligned} \tag{172}$$

$$\begin{aligned}\varphi_{10b}^{1,3,2k} = {} & \big(N_0/k^2\big)(675/4)^{1/2}\kappa_r\kappa_\rho^3 j_{1,3}(+)^\tau \\ & \times\big\{{}^4(\boldsymbol{\rho\rho})\big[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/3\big] \\ & -(1/2)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\big[(\mathbf{k}\cdot\boldsymbol{\rho})^2-k^2\rho^2/3\big] \\ & -{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\mathbf{r})\rho^2/5+{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\boldsymbol{\rho})\rho^2/5\big\}\end{aligned} \tag{173}$$

$$\begin{aligned}\varphi_{10c}^{3,1,2k} = {} & \big(N_0/k^2\big)(175/4)^{1/2}\kappa_r^3\kappa_\rho j_{3,1}(+)^\tau \\ & \times\big\{{}^4(\mathbf{rr})\big[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/7\big] \\ & +(1/2)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\big[(\mathbf{k}\cdot\mathbf{r})^2-k^2r^2/7\big] \\ & -(2/7)\,{}^4(\mathbf{kr}+\mathbf{rk})\big[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})+(\mathbf{k}\cdot\boldsymbol{\rho})r^2/2\big] \\ & -{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\mathbf{r})r^2/7+{}^4(\mathbf{kk})(\mathbf{r}\cdot\boldsymbol{\rho})\big(4r^2/35\big)\big\}\end{aligned} \tag{174}$$

$$\begin{aligned}\varphi_{10c}^{1,3,2k} = {} & \big(N_0/k^2\big)(175/4)^{1/2}\kappa_r\kappa_\rho^3 j_{1,3}(+)^\tau \\ & \times\big\{{}^4(\boldsymbol{\rho\rho})\big[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/7\big] \\ & +(1/2)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\big[(\mathbf{k}\cdot\boldsymbol{\rho})^2-k^2\rho^2/7\big] \\ & -(2/7)\,{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\big[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})+(\mathbf{k}\cdot\mathbf{r})\rho^2/2\big] \\ & -{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\boldsymbol{\rho})\rho^2/7+{}^4(\mathbf{kk})(\mathbf{r}\cdot\boldsymbol{\rho})\big(4\rho^2/35\big)\big\}\end{aligned} \tag{175}$$

$$\begin{aligned}\varphi_{10s}^{2,2,2k} = {} & \big(N_0/k^2\big)(135/2)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau \\ & \times{}^4(\mathbf{kk})\big[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/3\big]\end{aligned} \tag{176}$$

$$\varphi_{10a}^{2,2,2k} = (N_0/k^2)(3375/56)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau$$
$$\times \{{}^4(\mathbf{rr})[(\mathbf{k}\cdot\boldsymbol{\rho})^2 - k^2\rho^2/3] - {}^4(\boldsymbol{\rho\rho})[(\mathbf{k}\cdot\mathbf{r})^2 - k^2r^2/3]$$
$$-(2/5)\,{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})$$
$$+(2/5)\,{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})\} \tag{177}$$

$$\varphi_{10b}^{2,2,2k} = (N_0/k^2)(45/2)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau$$
$$\times [{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) - {}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})] \tag{178}$$

$$\varphi_{10c}^{2,2,2k} = (N_0/k^2)(6075/98)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau$$
$$\times \{{}^4(\mathbf{kr}+\mathbf{rk})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\mathbf{r})(2\rho^2/3)]$$
$$+{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\boldsymbol{\rho})(2r^2/3)]$$
$$-(2k^2/3)[{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - {}^4(\mathbf{rr})(2\rho^2/3) - {}^4(\boldsymbol{\rho\rho})(2r^2/3)]$$
$$-(4/3)\,{}^4(\mathbf{kk})[(\mathbf{r}\cdot\boldsymbol{\rho})^2 - 2r^2\rho^2/3]\} \tag{179}$$

$$\varphi_{10d}^{2,2,2k} = (N_0/k^2)(125/6)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau$$
$$\times \{{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho}) - (\mathbf{r}\cdot\boldsymbol{\rho})k^2/7]$$
$$+(1/2)\,{}^4(\mathbf{rr})[(\mathbf{k}\cdot\boldsymbol{\rho})^2 - k^2\rho^2/7]$$
$$+(1/2)\,{}^4(\boldsymbol{\rho\rho})[(\mathbf{k}\cdot\mathbf{r})^2 - k^2r^2/7]$$
$$-(2/7)\,{}^4(\mathbf{kr}+\mathbf{rk})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) + (\mathbf{k}\cdot\mathbf{r})\rho^2/2]$$
$$-(2/7)\,{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) + (\mathbf{k}\cdot\boldsymbol{\rho})r^2/2]$$
$$+(4/35)\,{}^4(\mathbf{kk})[(\mathbf{r}\cdot\boldsymbol{\rho})^2 + r^2\rho^2/2]\} \tag{180}$$

In addition to the last five functions, there is a sixth combination of terms which appears in the results of algebraic manipulations. This is the null combination given earlier in (7). This combination can be treated as though it were an expansion function, and then declared as a null function at the appropriate moment.

A few of the functions that are cubic in the momentum vector will also be listed here:

$$\varphi_{10}^{1,0,3k} = \left(iN_0/k^3\right)(30)^{1/2}\kappa_r j_{1,0}(-)^{\tau}$$

$$\times\left[{}^4(\mathbf{kk})(\mathbf{k}\cdot\mathbf{r}) - {}^4(\mathbf{kr}+\mathbf{rk})k^2/5\right] \tag{181}$$

$$\varphi_{10}^{0,1,3k} = \left(iN_0/k^3\right)(30)^{1/2}\kappa_\rho j_{0,1}(+)^{\tau}$$

$$\times\left[{}^4(\mathbf{kk})(\mathbf{k}\cdot\boldsymbol{\rho}) - {}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})k^2/5\right] \tag{182}$$

$$\varphi_{10}^{3,0,3k} = \left(iN_0/k^3\right)(875/8)^{1/2}\kappa_r^3 j_{3,0}(-)^{\tau}$$

$$\times\left\{{}^4(\mathbf{kr}+\mathbf{rk})\left[(\mathbf{k}\cdot\mathbf{r})^2 - k^2r^2/25\right]\right.$$

$$\left. -{}^4(\mathbf{rr})(\mathbf{k}\cdot\mathbf{r})\left(4k^2/5\right) - {}^4(\mathbf{kk})(\mathbf{k}\cdot\mathbf{r})\left(4r^2/5\right)\right\} \tag{183}$$

$$\varphi_{10}^{0,3,3k} = \left(iN_0/k^3\right)(875/8)^{1/2}\kappa_\rho^3 j_{0,3}(+)^{\tau}$$

$$\times\left\{{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\left[(\mathbf{k}\cdot\boldsymbol{\rho})^2 - k^2\rho^2/25\right]\right.$$

$$\left. -{}^4(\boldsymbol{\rho}\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho})\left(4k^2/5\right) - {}^4(\mathbf{kk})(\mathbf{k}\cdot\boldsymbol{\rho})\left(4\rho^2/5\right)\right\} \tag{184}$$

$$\varphi_{10s}^{2,1,3k} = \left(iN_0/k^3\right)(135)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^{\tau}$$

$$\times\left\{{}^4(\mathbf{kk})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\boldsymbol{\rho})r^2/3\right]\right.$$

$$\left. -{}^4(\mathbf{kr}+\mathbf{rk})(\mathbf{r}\cdot\boldsymbol{\rho})k^2/5 + {}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})k^2r^2/15\right\} \tag{185}$$

$$\varphi_{10s}^{1,2,3k} = \left(iN_0/k^3\right)(135)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^{\tau}$$

$$\times\left\{{}^4(\mathbf{kk})\left[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\mathbf{r})\rho^2/3\right]\right.$$

$$\left. -{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})(\mathbf{r}\cdot\boldsymbol{\rho})k^2/5 + {}^4(\mathbf{kr}+\mathbf{rk})k^2\rho^2/15\right\} \tag{186}$$

$$\varphi_{10b}^{2,1,3k} = \left(iN_0/k^3\right)(75/2)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^{\tau}$$

$$\times\left\{{}^4(\mathbf{kk})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - (\mathbf{k}\cdot\boldsymbol{\rho})r^2\right]\right.$$

$$+{}^4(\mathbf{kr}+\mathbf{rk})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho}) - (\mathbf{r}\cdot\boldsymbol{\rho})\left(2k^2/5\right)\right]$$

$$-{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\left[(\mathbf{k}\cdot\mathbf{r})^2-2k^2r^2/5\right]$$

$$-{}^4(\mathbf{rr})(\mathbf{k}\cdot\boldsymbol{\rho})(2k^2/5)+{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\mathbf{r})k^2/5\Big\} \tag{187}$$

$$\varphi_{10b}^{1,2,3k}=\left(iN_0/k^3\right)(75/2)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$

$$\times\Big\{{}^4(\mathbf{kk})\left[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})\rho^2\right]$$

$$+{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})(2k^2/5)\right]$$

$$-{}^4(\mathbf{kr}+\mathbf{rk})\left[(\mathbf{k}\cdot\boldsymbol{\rho})^2-2k^2\rho^2/5\right]$$

$$-{}^4(\boldsymbol{\rho}\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})(2k^2/5)+{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})k^2/5\Big\} \tag{188}$$

$$\varphi_{10c}^{2,1,3k}=\left(iN_0/k^3\right)(125/2)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau$$

$$\times\Big\{{}^4(\mathbf{kr}+\mathbf{rk})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/25\right]$$

$$+(1/2)\,{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\left[(\mathbf{k}\cdot\mathbf{r})^2-k^2r^2/25\right]$$

$$-(2/5)\,{}^4(\mathbf{rr})(\mathbf{k}\cdot\boldsymbol{\rho})k^2-(2/5)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\mathbf{r})k^2$$

$$-(4/5)\,{}^4(\mathbf{kk})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})+(\mathbf{k}\cdot\boldsymbol{\rho})r^2/2\right]\Big\} \tag{189}$$

$$\varphi_{10c}^{1,2,3k}=\left(iN_0/k^3\right)(125/2)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$

$$\times\Big\{{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\left[(\mathbf{k}\cdot\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-(\mathbf{r}\cdot\boldsymbol{\rho})k^2/25\right]$$

$$+(1/2)\,{}^4(\mathbf{kr}+\mathbf{rk})\left[(\mathbf{k}\cdot\boldsymbol{\rho})^2-k^2\rho^2/25\right]$$

$$-(2/5)\,{}^4(\boldsymbol{\rho}\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})k^2-(2/5)\,{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})k^2$$

$$-(4/5)\,{}^4(\mathbf{kk})\left[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})+(\mathbf{k}\cdot\mathbf{r})\rho^2/2\right]\Big\} \tag{190}$$

The above are even-parity ${}^4D$ functions constructed from (139)–(141) using the operators in (8)–(10). When we use these same operators on the odd-parity ${}^4D$ functions given in (142) and (143), we generate momentum-dependent odd-parity functions, some of which will be listed here.

The leading functions are the following six:

$$\varphi_{15s}^{1,1,k}=(N_0/k)(18/5)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^{\tau\,4}(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r}) \quad (191)$$

$$\varphi_{15a}^{1,1,k}=(N_0/k)(54/5)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^{\tau\,4}(i\mathbf{kr}\times\boldsymbol{\rho}) \quad (192)$$

$$\varphi_{15}^{2,0,k}=(N_0/k)(12)^{1/2}\kappa_r^2 j_{2,0}(-)^{\tau\,4}(i\mathbf{rk}\times\mathbf{r}) \quad (193)$$

$$\varphi_{15}^{0,2,k}=(N_0/k)(12)^{1/2}\kappa_\rho^2 j_{0,2}(-)^{\tau\,4}(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho}) \quad (194)$$

$$\varphi_{15}^{1,0,2k}=(iN_0/k^2)(12)^{1/2}\kappa_r j_{1,0}(-)^{\tau\,4}(i\mathbf{kk}\times\mathbf{r}) \quad (195)$$

$$\varphi_{15}^{0,1,2k}=(iN_0/k^2)(12)^{1/2}\kappa_\rho j_{0,1}(+)^{\tau\,4}(i\mathbf{kk}\times\boldsymbol{\rho}) \quad (196)$$

Other functions linear in the momentum vector include

$$\varphi_{15s}^{3,1,k}=(N_0/k)(60)^{1/2}\kappa_r^3\kappa_\rho j_{3,1}(+)^\tau \times\left[{}^4(i\mathbf{rk}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})r^2/5\right] \quad (197)$$

$$\varphi_{15s}^{1,3,k}=(N_0/k)(60)^{1/2}\kappa_r\kappa_\rho^3 j_{1,3}(+)^\tau \times\left[{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})\rho^2/5\right] \quad (198)$$

$$\varphi_{15s}^{2,2,k}=(N_0/k)(270/7)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^\tau \times\left\{{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) -(2/3)\left[{}^4(i\mathbf{rk}\times\mathbf{r})\rho^2+{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})r^2\right]\right\} \quad (199)$$

$$\varphi_{15a}^{3,1,k}=(N_0/k)(135/2)^{1/2}\kappa_r^3\kappa_\rho j_{3,1}(+)^\tau \times\left[{}^4(i\mathbf{rr}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})+(1/3)\,{}^4(i\mathbf{rk}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) -(1/6)\,{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})r^2-(3/10)\,{}^4(i\mathbf{kr}\times\boldsymbol{\rho})r^2\right] \quad (200)$$

$$\varphi_{15a}^{1,3,k}=(N_0/k)(135/2)^{1/2}\kappa_r\kappa_\rho^3 j_{1,3}(+)^\tau \times\left[{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho})-(1/3)\,{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) +(1/6)\,{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})\rho^2-(3/10)\,{}^4(i\mathbf{kr}\times\boldsymbol{\rho})\rho^2\right] \quad (201)$$

$$\varphi_{15a}^{2,2,k}=(N_0/k)(54)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^{\tau\,4}(i\mathbf{kr}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) \tag{202}$$

$$\begin{aligned}\varphi_{15b}^{2,2,k}=&(N_0/k)(675/28)^{1/2}\kappa_r^2\kappa_\rho^2 j_{2,2}(-)^{\tau}\\&\times\{{}^4(i\mathbf{rr}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho})+{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})-(3/5)\,{}^4(i\mathbf{kr}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})\\&+(1/3)[{}^4(i\mathbf{rk}\times\mathbf{r})\rho^2-{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})r^2]\}\end{aligned} \tag{203}$$

In algebraic operations which generate the terms that appear in (199), (202), and (203), there will ordinarily be additional terms expressible as some multiple of the expression in (6). As noted earlier, that expression is a null quantity, a grouping which is identically zero. For convenience, it can be treated as belonging to a fictitious function $\varphi_{15c}^{2,2,k}$, which is then declared to be a null function.

In the case of the analogous null combination in (7), that grouping could have been treated as a factor in a fictitious expansion function $\varphi_{10e}^{2,2,2k}$ which used terms appearing in (176)–(180). That fictitious function could have been used in the algebraic sorting of terms, and then declared as a null function.

Further odd-parity ${}^4D$ functions, depending linearly on the momentum vector $\mathbf{k}$, are the following:

$$\begin{aligned}\varphi_{15s}^{4,2,k}=&(N_0/k)(525/2)^{1/2}\kappa_r^4\kappa_\rho^2 j_{4,2}(-)^{\tau}\\&\times\{{}^4(i\mathbf{rk}\times\mathbf{r})[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/7]\\&-(2/7)\,{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})r^2+(2/35)\,{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})r^4\}\end{aligned} \tag{204}$$

$$\begin{aligned}\varphi_{15s}^{2,4,k}=&(N_0/k)(525/2)^{1/2}\kappa_r^2\kappa_\rho^4 j_{2,4}(-)^{\tau}\\&\times\{{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/7]\\&-(2/7)\,{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})\rho^2+(2/35)\,{}^4(i\mathbf{rk}\times\mathbf{r})\rho^4\}\end{aligned} \tag{205}$$

$$\begin{aligned}\varphi_{15s}^{3,3,k}=&(N_0/k)(875/4)^{1/2}\kappa_r^3\kappa_\rho^3 j_{3,3}(+)^{\tau}\\&\times\{{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/25]\\&-(4/5)[{}^4(i\mathbf{rk}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})\rho^2+{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})r^2]\}\end{aligned} \tag{206}$$

$$\varphi_{15a}^{4,2,k} = (N_0/k)(945/2)^{1/2}\kappa_r^4\kappa_\rho^2 j_{4,2}(-)^\tau$$
$$\times\{{}^4(i\mathbf{rr}\times\boldsymbol{\rho})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})r^2/7]$$
$$+(1/3)\,{}^4(i\mathbf{rk}\times\mathbf{r})[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/7]$$
$$-(3/14)\,{}^4(i\mathbf{kr}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})r^2-(1/6)\,{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})r^2$$
$$-(1/7)\,{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})r^2+(1/21)\,{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})r^4\} \tag{207}$$

$$\varphi_{15a}^{2,4,k} = (N_0/k)(945/2)^{1/2}\kappa_r^2\kappa_\rho^4 j_{2,4}(-)^\tau$$
$$\times\{{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})\rho^2/7]$$
$$-(1/3)\,{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/7]$$
$$-(3/14)\,{}^4(i\mathbf{kr}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})\rho^2+(1/6)\,{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})\rho^2$$
$$-(1/7)\,{}^4(i\mathbf{rr}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho})\rho^2-(1/21)\,{}^4(i\mathbf{rk}\times\mathbf{r})\rho^4\} \tag{208}$$

$$\varphi_{15a}^{3,3,k} = (N_0/k)(945/4)^{1/2}\kappa_r^3\kappa_\rho^3 j_{3,3}(+)^\tau$$
$$\times{}^4(i\mathbf{kr}\times\boldsymbol{\rho})[(\mathbf{r}\cdot\boldsymbol{\rho})^2-r^2\rho^2/5] \tag{209}$$

$$\varphi_{15b}^{3,3,k} = (N_0/k)(875/4)^{1/2}\kappa_r^3\kappa_\rho^3 j_{3,3}(+)^\tau$$
$$\times\{{}^4(i\mathbf{rr}\times\boldsymbol{\rho})[(\mathbf{k}\cdot\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\mathbf{r})(2\rho^2/5)]$$
$$+{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})[(\mathbf{k}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-(\mathbf{k}\cdot\boldsymbol{\rho})(2r^2/5)]$$
$$-(3/5)\,{}^4(i\mathbf{kr}\times\boldsymbol{\rho})[(\mathbf{r}\cdot\boldsymbol{\rho})^3-2r^2\rho^2/5]$$
$$+(1/5)[{}^4(i\mathbf{rk}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})\rho^2-{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})r^2]\} \tag{210}$$

For convenience in algebraic manipulations, it may be appropriate here to define a fictitious function $\varphi_{15c}^{3,3,k}$ which will contain as a factor the expression in (6), multiplied by the further factor $(\mathbf{r}\cdot\boldsymbol{\rho})$. This function, analogous to $\varphi_{15c}^{2,2,k}$ discussed above, is then declared as a null function after it has been utilized to facilitate the sorting of the terms which appear in (206), (209), and (210).

Functions quadratic in the momentum vector include the following:

$$\varphi_{15}^{3,0,2k} = (iN_0/k^2)(75)^{1/2}\kappa_r^3 j_{3,0}(-)^\tau \times\left[{}^4(i\mathbf{rk}\times\mathbf{r})(\mathbf{k}\cdot\mathbf{r}) - {}^4(i\mathbf{kk}\times\mathbf{r})r^2/5\right] \tag{211}$$

$$\varphi_{15}^{0,3,2k} = (iN_0/k^2)(75)^{1/2}\kappa_\rho^3 j_{0,3}(+)^\tau \times\left[{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho}) - {}^4(i\mathbf{kk}\times\boldsymbol{\rho})\rho^2/5\right] \tag{212}$$

$$\begin{aligned}\varphi_{15s}^{2,1,2k} = {}&(iN_0/k^2)(75/7)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau \\ &\times\left[{}^4(i\mathbf{rk}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho}) + {}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\mathbf{r})\right.\\ &\left.-(2/5)\,{}^4(i\mathbf{kk}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) - (1/5)\,{}^4(i\mathbf{kk}\times\boldsymbol{\rho})r^2\right]\end{aligned} \tag{213}$$

$$\begin{aligned}\varphi_{15s}^{1,2,2k} = {}&(iN_0/k^2)(75/7)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau \\ &\times\left[{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r}) + {}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})\right.\\ &\left.-(2/5)\,{}^4(i\mathbf{kk}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) - (1/5)\,{}^4(i\mathbf{kk}\times\mathbf{r})\rho^2\right]\end{aligned} \tag{214}$$

$$\begin{aligned}\varphi_{15a}^{2,1,2k} = {}&(iN_0/k^2)(540/7)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau \\ &\times\left\{{}^4(i\mathbf{kr}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r}) + (2/3)\ {}^4(i\mathbf{rk}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})\right.\\ &-(1/3)\left[{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\mathbf{r})\right.\\ &\left.\left.-{}^4(i\mathbf{kk}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) + {}^4(i\mathbf{kk}\times\boldsymbol{\rho})r^2\right]\right\}\end{aligned} \tag{215}$$

$$\begin{aligned}\varphi_{15a}^{1,2,2k} = {}&(iN_0/k^2)(540/7)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau \\ &\times\left\{{}^4(i\mathbf{kr}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho}) - (2/3)\ {}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})\right.\\ &+(1/3)\left[{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})\right.\\ &\left.\left.-{}^4(i\mathbf{kk}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho}) + {}^4(i\mathbf{kk}\times\mathbf{r})\rho^2\right]\right\}\end{aligned} \tag{216}$$

$$\begin{aligned}\varphi_{15b}^{2,1,2k} = {}&(iN_0/k^2)(480/7)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau \\ &\times\left[{}^4(i\mathbf{rk}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho}) - (1/2)\,{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\mathbf{r})\right.\\ &\left.-(1/4)\,{}^4(i\mathbf{kk}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho}) + {}^4(i\mathbf{kk}\times\boldsymbol{\rho})r^2/4 + {}^4(i\mathbf{rr}\times\boldsymbol{\rho})k^2/2\right]\end{aligned} \tag{217}$$

$$\varphi_{15b}^{1,2,2k} = \left(iN_0/k^2\right)(480/7)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$
$$\times\left[{}^4(i\rho\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})-(1/2)\,{}^4(i\mathbf{r}\mathbf{k}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})\right.$$
$$\left.-(1/4)\,{}^4(i\mathbf{k}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})+{}^4(i\mathbf{k}\mathbf{k}\times\mathbf{r})\rho^2/4-{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})k^2/2\right] \tag{218}$$

$$\varphi_{15c}^{2,1,2k} = \left(iN_0/k^2\right)(54)^{1/2}\kappa_r^2\kappa_\rho j_{2,1}(+)^\tau$$
$$\times\left[{}^4(i\mathbf{k}\mathbf{k}\times\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{k}\mathbf{k}\times\boldsymbol{\rho})r^2/3\right] \tag{219}$$

$$\varphi_{15c}^{1,2,2k} = \left(iN_0/k^2\right)(54)^{1/2}\kappa_r\kappa_\rho^2 j_{1,2}(-)^\tau$$
$$\times\left[{}^4(i\mathbf{k}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{r}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{k}\mathbf{k}\times\mathbf{r})\rho^2/3\right] \tag{220}$$

A few of the functions cubic in the momentum vector are

$$\varphi_{15}^{2,0,3k} = \left(N_0/k^3\right)(75)^{1/2}\kappa_r^2 j_{2,0}(-)^\tau$$
$$\times\left[{}^4(i\mathbf{k}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\mathbf{r})-{}^4(i\mathbf{r}\mathbf{k}\times\mathbf{r})k^2/5\right] \tag{221}$$

$$\varphi_{15}^{0,2,3k} = \left(N_0/k^3\right)(75)^{1/2}\kappa_\rho^2 j_{0,2}(-)^\tau$$
$$\times\left[{}^4(i\mathbf{k}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\boldsymbol{\rho})-{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})k^2/5\right] \tag{222}$$

$$\varphi_{15s}^{1,1,3k} = \left(N_0/k^3\right)(45/2)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^\tau$$
$$\times\left[{}^4(i\mathbf{k}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})+{}^4(i\mathbf{k}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})\right.$$
$$\left.-{}^4(i\mathbf{r}\mathbf{k}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})k^2/5\right] \tag{223}$$

$$\varphi_{15a}^{1,1,3k} = \left(N_0/k^3\right)(45)^{1/2}\kappa_r\kappa_\rho j_{1,1}(+)^\tau$$
$$\times\left[{}^4(i\mathbf{k}\mathbf{k}\times\mathbf{r})(\mathbf{k}\cdot\boldsymbol{\rho})-{}^4(i\mathbf{k}\mathbf{k}\times\boldsymbol{\rho})(\mathbf{k}\cdot\mathbf{r})\right.$$
$$\left.+{}^4(i\mathbf{k}\mathbf{r}\times\boldsymbol{\rho})(3k^2/5)\right] \tag{224}$$

There are, of course, many more functions (infinitely many more). They can be generated as needed with the aid of the operators in (8)–(10), though in some cases considerable trial and error is needed before the generated terms can be correctly sorted into correct functions. Correct functions are those that make the matrix representations of these operators symmetric, while making the matrix representation of the operator in (11) skew symmetric. What we have given here are the earlier functions in what can be recognized as families and subfamilies, whose orderliness suggests that generalized formulas for the families must exist, similar to the generalized formulas in III, though we have not found those formulas as yet.

We have, however, found procedures for transliterating from the functions to the coefficients. These procedures were illustrated earlier, as they applied to the ^2S, ^4P, and ^2P functions and coefficients. With respect to the ^4D families, the added transliteration formulas are, for even-parity functions,

$$\begin{gathered}
{}^4(\mathbf{rr})\to(1/6)^{1/2}C_{10}\\
{}^4(\boldsymbol{\rho\rho})\to(1/6)^{1/2}C_{11}\\
{}^4(\mathbf{r}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{r})\to(5/9)^{1/2}C_{12}\\
{}^4(\mathbf{kr}+\mathbf{rk})\to-i(5/9)^{1/2}C_{10}^{1,0,k}\\
{}^4(\mathbf{k}\boldsymbol{\rho}+\boldsymbol{\rho}\mathbf{k})\to-i(5/9)^{1/2}C_{10}^{0,1,k}\\
{}^4(\mathbf{kk})\to(1/6)^{1/2}C_{10}^{0,0,2k}
\end{gathered}\tag{225}$$

For the odd-parity ^4D functions, we get

$$\begin{gathered}
{}^4(i\mathbf{rr}\times\boldsymbol{\rho})\to-i(1/12)^{1/2}C_{15}\\
{}^4(i\boldsymbol{\rho}\mathbf{r}\times\boldsymbol{\rho})\to-i(1/12)^{1/2}C_{16}\\
{}^4(i\mathbf{rk}\times\boldsymbol{\rho}+i\boldsymbol{\rho}\mathbf{k}\times\mathbf{r})\to(5/18)^{1/2}C_{15s}^{1,1,k}\\
{}^4(i\mathbf{kr}\times\boldsymbol{\rho})\to(5/54)^{1/2}C_{15a}^{1,1,k}\\
{}^4(i\mathbf{rk}\times\mathbf{r})\to(1/12)^{1/2}C_{15}^{2,0,k}\\
{}^4(i\boldsymbol{\rho}\mathbf{k}\times\boldsymbol{\rho})\to(1/12)^{1/2}C_{15}^{0,2,k}\\
{}^4(i\mathbf{kk}\times\mathbf{r})\to-i(1/12)^{1/2}C_{15}^{1,0,2k}\\
{}^4(i\mathbf{kk}\times\boldsymbol{\rho})\to-i(1/12)^{1/2}C_{15}^{0,1,2k}
\end{gathered}\tag{226}$$

Other quantities are transliterated as shown in (98) and (99), and in the discussion that follows those specifications.

Examples of transliterated coefficients for ^4D functions are

$$C_{10s}^{1,1,2k} = -(3)^{1/2}\nu C_{10}^{0,0,2k} \tag{227}$$

$$C_{10b}^{2,2,2k} = (25/2)^{1/2}[-\nu\nu'' C_{10}^{1,0,k} + \nu\nu' C_{10}^{0,1,k}] \tag{228}$$

$$C_{15s}^{1,3,k} = -(5)^{1/2}\nu C_{15}^{0,2,k} + (2/3)^{1/2} C_{15s}^{1,1,k} \tag{229}$$

$$C_{15c}^{2,1,2k} = (9/2)^{1/2}[-\nu C_{15}^{1,0,2k} + (1/3) C_{15}^{0,1,2k}] \tag{230}$$

These transliterations arise out of the auxiliary equations (8)–(10). There is an interplay between action by the differential operators upon the expansion functions and matrix manipulations of the expansion coefficients. This interplay is embodied in and summarized by the transliterations.

Consider now the null functions obtainable from (6) and (7). These identities express a lack of full independence among the functions contained therein. Yet the transliteration formulas will permit the construction of putative expansion coefficients to accompany these null functions (except for an uncertainty as to the numerical normalization factors). Each such coefficient will be a linear combination of leading coefficients. We can tentatively infer that the lack of full independence among the functions means a corresponding lack of full independence among the coefficients, expressed then by the two null conditions:

$$0 = \nu'' C_{15} - \nu' C_{16} - C_{15}^{2,0,k} - C_{15}^{0,2,k} + (10/3)^{1/2}\nu C_{15s}^{1,1,k} \tag{231}$$

$$\begin{aligned} 0 = {} & (\nu''^2 - 1) C_{10} + (\nu'^2 - 1) C_{11} - (\nu^2 - 1) C_{10}^{0,0,2k} \\ & - (10/3)^{1/2}[(\nu'\nu'' - \nu) C_{12} + (\nu\nu'' - \nu') C_{10}^{1,0,k} \\ & + (\nu\nu' - \nu'') C_{10}^{0,1,k}] \end{aligned} \tag{232}$$

The relationships (231) and (232) will be found to be consistent with other algebraic conditions, or to follow from other algebraic conditions, though here they are just presented as tentative inferences from the transliteration formulas, together with the easily verifiable identities (6) and (7). The transliteration formulas themselves were obtained inductively, through inspection of the first few dozen coefficients constructed by explicit application of the auxiliary equations (8)–(10). A formal inductive proof of these transliteration formulas will be left to the reader.

These transliteration formulas permit us to write down expressions for any of the ^4D coefficients, in terms of the leading coefficients, where these latter are six in number for the even-parity family and eight in number for the odd-parity family. The relationships (231) and (232) reduce these numbers to five and seven. We will want to reduce the number of independent leading coefficients still further, through use of the auxiliary condition (11) which couples together the even-parity and odd-parity families.

In analogy with the ${}^4P$ relationships in (101)–(106), we can construct, for the ${}^4D$ families, the relationships

$$\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{10}=(1/15)^{1/2}\varphi_{15s}^{1,1,k}+(1/45)^{1/2}\varphi_{15a}^{1,1,k}+(2/45)^{1/2}\varphi_{15s}^{3,1,k}+(4/45)^{1/2}\varphi_{15a}^{3,1,k} \tag{233}$$

$$\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{11}=-(1/15)^{1/2}\varphi_{15s}^{1,1,k}+(1/45)^{1/2}\varphi_{15a}^{1,1,k}-(2/45)^{1/2}\varphi_{15s}^{1,3,k}+(4/45)^{1/2}\varphi_{15a}^{1,3,k} \tag{234}$$

$$\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{12}=-(1/15)^{1/2}\varphi_{15}^{2,0,k}+(1/15)^{1/2}\varphi_{15}^{0,2,k}-(2/375)^{1/2}\varphi_{15a}^{2,2,k}+(28/375)^{1/2}\varphi_{15b}^{2,2,k} \tag{235}$$

$$\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{10}^{1,0,k}=(1/15)^{1/2}\varphi_{15}+(1/15)^{1/2}\varphi_{15}^{0,1,2k}+(7/1200)^{1/2}\varphi_{15a}^{2,1,2k}+(7/600)^{1/2}\varphi_{15b}^{2,1,2k}+(1/30)^{1/2}\varphi_{15c}^{2,1,2k} \tag{236}$$

$$\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{10}^{0,1,k}=(1/15)^{1/2}\varphi_{16}-(1/15)^{1/2}\varphi_{15}^{1,0,2k}+(7/1200)^{1/2}\varphi_{15a}^{1,2,2k}-(7/600)^{1/2}\varphi_{15b}^{1,2,2k}-(1/30)^{1/2}\varphi_{15c}^{1,2,2k} \tag{237}$$

$$\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{10}^{0,0,2k}=(4/45)^{1/2}\varphi_{15a}^{1,1,k}+(2/15)^{1/2}\varphi_{15a}^{1,1,3k} \tag{238}$$

$$\begin{aligned}\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{15}=&-(1/5)^{1/2}\varphi_{10}^{1,0,k}\\&-(2/45)^{1/2}\varphi_{10}^{3,0,k}-(2/375)^{1/2}\varphi_{10s}^{1,2,k}\\&+(8/225)^{1/2}\varphi_{10b}^{1,2,k}-(14/1125)^{1/2}\varphi_{10c}^{1,2,k}\\&+(1/125)^{1/2}\varphi_{10s}^{3,2,k}-(2/225)^{1/2}\varphi_{10b}^{3,2,k}\\&-(32/375)^{1/2}\varphi_{10c}^{3,2,k}\end{aligned} \tag{239}$$

$$\begin{aligned}\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{16}=&-(1/15)^{1/2}\varphi_{10}^{0,1,k}\\&-(2/45)^{1/2}\varphi_{10}^{0,3,k}-(2/375)^{1/2}\varphi_{10s}^{2,1,k}\\&+(8/225)^{1/2}\varphi_{10b}^{2,1,k}-(14/1125)^{1/2}\varphi_{10c}^{2,1,k}\\&+(1/125)^{1/2}\varphi_{10s}^{2,3,k}-(2/225)^{1/2}\varphi_{10b}^{2,3,k}\\&-(32/375)^{1/2}\varphi_{10c}^{2,3,k}\end{aligned} \tag{240}$$

$$\begin{aligned}\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{15}^{2,0,k}=&(1/15)^{1/2}\varphi_{12}\\&+(2/45)^{1/2}\varphi_{26}-(1/150)^{1/2}\varphi_{10b}^{1,1,2k}\\&-(7/150)^{1/2}\varphi_{10c}^{1,1,2k}-(7/225)^{1/2}\varphi_{10s}^{3,1,2k}\\&-(4/15)\varphi_{10b}^{3,1,2k}\end{aligned} \tag{241}$$

$$\begin{aligned}\left(P^{\tau}/k\kappa_r\kappa_\rho\right)\left(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho\right)\varphi_{15}^{0,2,k}=&-(1/15)^{1/2}\varphi_{12}\\&-(2/45)^{1/2}\varphi_{27}-(1/150)^{1/2}\varphi_{10b}^{1,1,2k}\\&+(7/150)^{1/2}\varphi_{10c}^{1,1,2k}+(7/225)^{1/2}\varphi_{10s}^{1,3,2k}\\&+(4/15)\varphi_{10b}^{1,3,2k}\end{aligned} \tag{242}$$

$$\begin{aligned}(P^{\tau}/k\kappa_r\kappa_\rho)(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\varphi_{15s}^{1,1,k} &= -(1/15)^{1/2}\varphi_{10}\\ &\quad+(1/15)^{1/2}\varphi_{11}+(7/150)^{1/2}\varphi_{10}^{2,0,2k}\\ &\quad-(7/150)^{1/2}\varphi_{10}^{0,2,2k}-(112/1875)^{1/2}\varphi_{10a}^{2,2,2k}\\ &\quad+(1/25)\varphi_{10b}^{2,2,2k} \qquad (243)\end{aligned}$$

$$\begin{aligned}(P^{\tau}/k\kappa_r\kappa_\rho)(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\varphi_{15a}^{1,1,k} &= -(1/45)^{1/2}\varphi_{10}\\ &\quad-(1/45)^{1/2}\varphi_{11}-(14/225)^{1/2}\varphi_{28}-(4/45)^{1/2}\varphi_{10}^{0,0,2k}\\ &\quad-(7/450)^{1/2}\varphi_{10}^{2,0,2k}-(7/450)^{1/2}\varphi_{10}^{0,2,2k}\\ &\quad+(2/15)\varphi_{10s}^{2,2,2k}-(49/1125)^{1/2}\varphi_{10c}^{2,2,2k} \qquad (244)\end{aligned}$$

$$\begin{aligned}(P^{\tau}/k\kappa_r\kappa_\rho)(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\varphi_{15}^{1,0,2k} &= (1/15)^{1/2}\varphi_{10}^{0,1,k}\\ &\quad+(1/30)^{1/2}\varphi_{10s}^{2,1,k}-(1/50)^{1/2}\varphi_{10b}^{2,1,k}-(2/45)^{1/2}\varphi_{10}^{0,1,3k}\\ &\quad-(1/45)^{1/2}\varphi_{10s}^{2,1,3k}-(2/25)^{1/2}\varphi_{10b}^{2,1,3k} \qquad (245)\end{aligned}$$

$$\begin{aligned}(P^{\tau}/k\kappa_r\kappa_\rho)(i\mathbf{k}\cdot\nabla_r\times\nabla_\rho)\varphi_{15}^{0,1,2k} &= -(1/15)^{1/2}\varphi_{10}^{1,0,k}\\ &\quad-(1/30)^{1/2}\varphi_{10s}^{1,2,k}+(1/50)^{1/2}\varphi_{10b}^{1,2,k}+(2/45)^{1/2}\varphi_{10}^{1,0,3k}\\ &\quad+(1/45)^{1/2}\varphi_{10s}^{1,2,3k}+(2/25)^{1/2}\varphi_{10b}^{1,2,3k} \qquad (246)\end{aligned}$$

As noted earlier, the operator on the left of (233)–(246) is skew symmetric, and there are many illustrations of this among the matrix elements on the right. Substitution into the eigenvalue equation (11) leads to relationships among the expansion coefficients. The relationship that corresponds to (233) is

$$\begin{aligned}\gamma C_{10} &= (1/15)^{1/2}C_{15s}^{1,1,k}+(1/45)^{1/2}C_{15a}^{1,1,k}\\ &\quad+(2/45)^{1/2}C_{15s}^{3,1,k}+(4/45)^{1/2}C_{15a}^{3,1,k} \qquad (247)\end{aligned}$$

and similar relationships can be directly obtained from the 13 others.

Each of the 14 relationships among coefficients can then be reduced through substitutions such as

$$C_{15s}^{3,1,k} = -(5)^{1/2}\nu C_{15}^{2,0,k} + (2/3)^{1/2} C_{15s}^{1,1,k} \tag{248}$$

$$C_{15a}^{3,1,k} = (45/8)^{1/2}\big[\nu' C_{15} - (1/3)\nu C_{15}^{2,0,k} + (5/54)^{1/2} C_{15s}^{1,1,k} + (1/10)^{1/2} C_{15a}^{1,1,k}\big] \tag{249}$$

What results is then the following set of 14 linear homogeneous equations, in the 14 leading coefficients:

$$\gamma C_{10} = (1/2)^{1/2}\big[\nu' C_{15} - \nu C_{15}^{2,0,k} + (5/6)^{1/2} C_{15s}^{1,1,k} + (5/18)^{1/2} C_{15a}^{1,1,k}\big] \tag{250}$$

$$\gamma C_{11} = (1/2)^{1/2}\big[\nu'' C_{16} + \nu C_{15}^{0,2,k} - (5/6)^{1/2} C_{15s}^{1,1,k} + (5/18)^{1/2} C_{15a}^{1,1,k}\big] \tag{251}$$

$$\gamma C_{12} = (3/20)^{1/2}\big[\nu'' C_{15} + \nu' C_{16} + (10/9)^{1/2}\nu C_{15a}^{1,1,k} - C_{15}^{2,0,k} + C_{15}^{0,2,k}\big] \tag{252}$$

$$\gamma C_{10}^{1,0,k} = (3/20)^{1/2}\big[C_{15} + (5/6)^{1/2}\nu' C_{15s}^{1,1,k} - (5/18)^{1/2}\nu' C_{15a}^{1,1,k} - \nu'' C_{15}^{2,0,k} - \nu C_{15}^{1,0,2k} + C_{15}^{0,1,2k}\big] \tag{253}$$

$$\gamma C_{10}^{0,1,k} = (3/20)^{1/2}\big[C_{16} - (5/6)^{1/2}\nu'' C_{15s}^{1,1,k} - (5/18)^{1/2}\nu'' C_{15a}^{1,1,k} + \nu' C_{15}^{0,2,k} - C_{15}^{1,0,2k} + \nu C_{15}^{0,1,2k}\big] \tag{254}$$

$$\gamma C_{10}^{0,0,2k} = (1/2)^{1/2}\big[(10/9)^{1/2} C_{15a}^{1,1,k} + \nu'' C_{15}^{1,0,2k} - \nu' C_{15}^{0,1,2k}\big] \tag{255}$$

$$\gamma C_{15} = (2)^{1/2}(-\nu\nu'' + \nu')C_{10} + (5/3)^{1/2}\big[(-\nu\nu' + \nu'')C_{12} + (\nu^2 - 1)C_{10}^{1,0,k}\big] \tag{256}$$

$$\gamma C_{16}=(2)^{1/2}(-\nu\nu'+\nu'')C_{11}$$
$$+(5/3)^{1/2}\left[(-\nu\nu''+\nu')C_{12}+(\nu^2-1)C_{10}^{0,1,k}\right] \tag{257}$$

$$\gamma C_{15s}^{1,1,k}=(3/5)^{1/2}\left[(\nu''^2-1)C_{10}-(\nu'^2-1)C_{11}\right]$$
$$+(1/2)^{1/2}\left[(-\nu\nu''+\nu')C_{10}^{1,0,k}+(\nu\nu'-\nu'')C_{10}^{0,1,k}\right] \tag{258}$$

$$\gamma C_{15a}^{1,1,k}=(3/2)^{1/2}\left[(\nu\nu''-\nu')C_{10}^{1,0,k}+(\nu\nu'-\nu'')C_{10}^{0,1,k}\right]$$
$$+(9/5)^{1/2}(\nu^2-1)C_{10}^{0,0,2k} \tag{259}$$

$$\gamma C_{15}^{2,0,k}=(2)^{1/2}(\nu'\nu''-\nu)C_{10}$$
$$+(5/3)^{1/2}\left[(-\nu'^2+1)C_{12}+(\nu\nu'-\nu'')C_{10}^{1,0,k}\right] \tag{260}$$

$$\gamma C_{15}^{0,2,k}=(2)^{1/2}(-\nu'\nu''+\nu)C_{11}$$
$$+(5/3)^{1/2}\left[(\nu''^2-1)C_{12}-(\nu\nu''-\nu')C_{10}^{0,1,k}\right] \tag{261}$$

$$\gamma C_{15}^{1,0,2k}=(5/3)^{1/2}\left[(\nu'\nu''-\nu)C_{10}^{1,0,k}-(\nu'^2-1)C_{10}^{0,1,k}\right]$$
$$+(2)^{1/2}(-\nu\nu'+\nu'')C_{10}^{0,0,2k} \tag{262}$$

$$\gamma C_{15}^{0,1,2k}=(5/3)^{1/2}\left[(\nu''^2-1)C_{10}^{1,0,k}-(\nu'\nu''-\nu)C_{10}^{0,1,k}\right]$$
$$+(2)^{1/2}(\nu\nu''-\nu')C_{10}^{0,0,2k} \tag{263}$$

These 14 equations can immediately be used to verify the earlier tentative relationships (231) and (232). When the linear combination of (256)–(258) and (261), (262) is formed which will give, on its left, the grouping in (231), it is found that the result is an equation with exactly zero on the right-hand side of the equals sign. A similar grouping of (250)–(255) verifies (232).

The compatibility of the set of linear homogeneous equations would ordinarily depend on the vanishing of a secular determinant. However, in this instance no new condition is obtained, just the old condition (13), which was already established algebraically.

By judicious substitutions, using (231) and (232) and (250)–(263), we can construct the following reduction formulas:

$$C_{10}^{1,0,k}=(1-\nu^2)^{-1}\left[(6/5)^{1/2}(-\nu\nu''+\nu')C_{10}+(-\nu\nu'+\nu'')C_{12}-(3/5)^{1/2}\gamma C_{15}\right] \tag{264}$$

$$C_{10}^{0,1,k}=(1-\nu^2)^{-1}\left[(6/5)^{1/2}(-\nu\nu'+\nu'')C_{11}+(-\nu\nu''+\nu')C_{12}-(3/5)^{1/2}\gamma C_{16}\right] \tag{265}$$

$$\begin{aligned}C_{10}^{0,0,2k}={}&-2\gamma^2(1-\nu^2)^{-2}\left[C_{10}+C_{11}-(10/3)^{1/2}\nu C_{12}\right]\\&+(1-\nu^2)^{-1}\left[(\nu''^2-1)C_{10}+(\nu'^2-1)C_{11}-(10/3)^{1/2}(\nu'\nu''-\nu)C_{12}\right]\\&+(2)^{1/2}\gamma(1-\nu^2)^{-2}\left[(-\nu\nu''+\nu')C_{15}+(-\nu\nu'+\nu'')C_{16}\right]\end{aligned} \tag{266}$$

$$C_{15s}^{1,1,k}=(1-\nu^2)^{-1}\left[(3/5)^{1/2}\gamma(C_{10}-C_{11})+(3/10)^{1/2}(\nu\nu''-\nu')C_{15}+(3/10)^{1/2}(-\nu\nu'+\nu'')C_{16}\right] \tag{267}$$

$$\begin{aligned}C_{15a}^{1,1,k}={}&(9/5)^{1/2}\gamma(1-\nu^2)^{-1}\left[C_{10}+C_{11}-(10/3)^{1/2}\nu C_{12}\right]\\&+(9/10)^{1/2}(1-\nu^2)^{-1}\left[(\nu\nu''-\nu')C_{15}+(\nu\nu'-\nu'')C_{16}\right]\end{aligned} \tag{268}$$

$$C_{15}^{2,0,k}=(1-\nu^2)^{-1}\left[(2)^{1/2}\gamma\nu C_{10}-(5/3)^{1/2}\gamma C_{12}+(-\nu\nu'+\nu'')C_{15}\right] \tag{269}$$

$$C_{15}^{0,2,k}=(1-\nu^2)^{-1}\left[-(2)^{1/2}\gamma\nu C_{11}+(5/3)^{1/2}\gamma C_{12}+(\nu\nu''-\nu')C_{16}\right] \tag{270}$$

$$\begin{aligned}C_{15}^{1,0,2k}={}&(8)^{1/2}\gamma(\nu\nu'-\nu'')(1-\nu^2)^{-2}\left[C_{10}+C_{11}-(10/3)^{1/2}\nu C_{12}\right]\\&+(2)^{1/2}\gamma(1-\nu^2)^{-1}\left[\nu''C_{10}-(5/6)^{1/2}\nu'C_{12}\right]\end{aligned}$$

$$+2(\nu\nu'-\nu'')(1-\nu^2)^{-2}[(\nu\nu''-\nu')C_{15}+(\nu\nu'-\nu'')C_{16}]$$
$$+(1-\nu^2)^{-1}[(-\nu'\nu''+\nu)C_{15}+(\nu''^2-1)C_{16}] \tag{271}$$

$$\begin{aligned}C_{15}^{0,1,2k} &= (8)^{1/2}\gamma(-\nu\nu''+\nu')(1-\nu^2)^{-2}\\ &\quad\times[C_{10}+C_{11}-(10/3)^{1/2}\nu C_{12}]\\ &\quad+(2)^{1/2}\gamma(1-\nu^2)^{-1}[-\nu'C_{11}+(5/6)^{1/2}\nu''C_{12}]\\ &\quad+2(-\nu\nu''+\nu')(1-\nu^2)^{-2}\\ &\quad\times[(\nu\nu''-\nu')C_{15}+(\nu\nu'-\nu'')C_{16}]\\ &\quad+(1-\nu^2)^{-1}[(-\nu''^2+1)C_{15}+(\nu'\nu''-\nu)C_{16}]\end{aligned} \tag{272}$$

With these reduction formulas, we can finally express any ${}^4D$ coefficient as a linear combination of the five leading rest-system coefficients: C_{10}, C_{11}, C_{12}, C_{15}, and C_{16}.

8. DISCUSSION

In the previous article, III, we described the rest-system functions in the tree expansion. Generalized forms were given for those functions, permitting the infinite system to be compactly presented. With one auxiliary condition, and with one auxiliary parameter, ν, it became possible to express any expansion coefficient in terms of the first 16, $C_1, C_2, \ldots, C_{16}$, as a linear combination in which ν entered as a parameter.

In the present article the tree expansion has been extended to include the functions which lie outside the rest system. These momentum-dependent functions have been individually constructed, and show many familial relationships, but we have not been able as yet to find generalized formulas which will summarize the members of the families. Clearly what we are seeing are generalizations of spherical harmonics, and vector spherical harmonics, into a space of higher dimensionality than the space we are familiar with.

While we do not have a generalized Rodrigues formula, we do have the functions themselves—enough of them for the immediate needs. We have the auxiliary conditions (8)–(11) through which further functions can be generated as may be needed at a later time. Furthermore, we have a procedure by which the expansion coefficients accompanying these momentum-dependent functions can all be expressed as linear combinations of the first 16 rest-system coefficients. There are now four auxiliary eigenvalues, ν, ν', ν'', and γ, which enter as parameters in these expressions.

This permits us (as will be seen in the next article) to replace infinite matrix equations by finite sets of coupled equations, linear homogeneous equations in 16 unknowns, the 16 leading rest-system coefficients $C_1, \ldots, C_{16}$. Requiring that these coupled equations be compatible places restrictions on the parameters of the system, and in particular on the mass to be associated with a trilocal structure.

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